# Lie Groups: A Primer for a Particle Physicist

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## 1 Basics on Lie Groups

Intuitively, the $n$-parameteric Lie group $G$ is an infinite group with elements that are continuously parameterizable by $n$ real parameters $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n)$ in at least a finite neighbourhood of a unit element $e \in G$. In the neighbourhood the parameterization of $g(\vec{\alpha}) \in G$ can be chosen in such a way that

$$g(\vec{0}) = e. \tag{1}$$

The group composition law is defined by $n$ functions $(f_1, \ldots, f_n) \equiv \vec{f}$ so that

$$g(\vec{\alpha})g(\vec{\beta}) = g(\vec{f}(\vec{\alpha}, \vec{\beta})). \tag{2}$$

The functions $f_a(\vec{\alpha}, \vec{\beta})$ are required to be expandable to the Taylor series in $\vec{\alpha} = \vec{\beta} = \vec{0}$. We expect the composition function $\vec{f}$ to fulfill the following conditions

$$\vec{f}(\vec{0}, \vec{0}) = \vec{0}, \tag{3}$$

$$\vec{f}(\vec{\alpha}, \vec{0}) = \vec{\alpha}, \qquad \vec{f}(\vec{0}, \vec{\beta}) = \vec{\beta}, \tag{4}$$

which are equivalent to the identities $ee = e$, $g(\vec{\alpha})e = g(\vec{\alpha})$, and $eg(\vec{\beta}) = g(\vec{\beta})$, respectively. In addition, it seems natural to require

$$\vec{f}(\vec{\alpha}, \vec{\alpha}) = 2\vec{\alpha}, \tag{5}$$

which means that $g(\vec{\alpha})g(\vec{\alpha}) = g(2\vec{\alpha})$.

Let us expand the function $\vec{f}$ in the Taylor series at $\vec{\alpha} = \vec{\beta} = \vec{0}$

$$\begin{aligned} f_a(\vec{\alpha}, \vec{\beta}) = {} & f_a(\vec{0}, \vec{0}) + \left.\frac{\partial f_a}{\partial \alpha_b}\right|_{(0,0)} \cdot \alpha_b + \left.\frac{\partial f_a}{\partial \beta_b}\right|_{(0,0)} \cdot \beta_b \\ & + \frac{1}{2!} \left.\frac{\partial^2 f_a}{\partial \alpha_b \partial \alpha_c}\right|_{(0,0)} \cdot \alpha_b \alpha_c + \underbrace{\left.\frac{\partial^2 f_a}{\partial \alpha_b \partial \beta_c}\right|_{(0,0)}}_{C_{abc}} \cdot \alpha_b \beta_c + \frac{1}{2!} \left.\frac{\partial^2 f_a}{\partial \beta_b \partial \beta_c}\right|_{(0,0)} \cdot \beta_b \beta_c + \ldots, \end{aligned} \tag{6}$$

where the dots indicate higher-order terms. Taking into account eqs. (3) and (4) we obtain

$$\left.\frac{\partial f_a}{\partial \alpha_b}\right|_{(0,0)} = 1, \qquad \left.\frac{\partial^n f_a}{\partial \alpha_{b_1} \ldots \partial \alpha_{b_n}}\right|_{(0,0)} = 0, \ \forall n > 1, \tag{7}$$

$$\left.\frac{\partial f_a}{\partial \beta_b}\right|_{(0,0)} = 1, \qquad \left.\frac{\partial^n f_a}{\partial \beta_{b_1} \ldots \partial \beta_{b_n}}\right|_{(0,0)} = 0, \ \forall n > 1. \tag{8}$$

Then the Taylor expansion of $f_a(\vec{\alpha}, \vec{\beta})$ reads

$$f_a(\vec{\alpha}, \vec{\beta}) = \alpha_a + \beta_a + C_{abc}\alpha_b\beta_c + \ldots \text{(mixed terms of higher order)} \ldots, \tag{9}$$

where

$$C_{abc} = -C_{acb}, \tag{10}$$

due to (5). All this means that $g(\vec{\alpha})g(\vec{\beta}) = g(\vec{\alpha} + \vec{\beta} + \textit{mixed terms of higher order})$.

Let us express an element $g(\vec{\varepsilon}) \in G$ infinitesimally close to the unit element in the following form

$$g(\vec{\varepsilon}) = e + i\varepsilon_a J_a, \qquad \varepsilon_a \to 0. \tag{11}$$

where $J_a$'s ($a = 1, \dots, n$) are called the *generators*[1] of the group and the factor "$i$" in the linear term is a matter of convention. Without a proof we assume that the objects $J_a$ with appropriate qualities do exist and thus it makes sense to write down the expression (11). By composing any two infinitesimal elements $g(\vec{\varepsilon}_1), g(\vec{\varepsilon}_2) \in G$ we get an infinitesimal element of $G$ again. Since $g(\vec{\varepsilon}_1)g(\vec{\varepsilon}_2) = e + i(\vec{\varepsilon}_{1a} + \vec{\varepsilon}_{2a})J_a$ any linear combination of $J_a$'s with infinitesimal coefficients generates an infinitesimal element of $G$. The generators $J_a$ can be taken as a basis of an $n$-dimensional real vector space. Rephrasing the previous statement, any infinitesimal vector of the vector space generates an infinitesimal element of $G$. It also implies that the choice of the generators, and consequently the parameterization, of a given group is not unique. We are free to choose as generators any basis of the vector space.

Repeating the infinitesimal transformation (11) $N$-times while substituting $\varepsilon_a = \alpha_a/N$ where $N$ is an integer we obtain

$$\begin{aligned} g(\vec{\alpha}) &= \left(\mathbf{I} + i\frac{\alpha_a}{N}J_a\right)^N = \sum_{k=0}^{N} \begin{pmatrix} N \\ k \end{pmatrix} \left(i\frac{\alpha_a}{N}J_a\right)^{N-k} \\ &= \mathbf{I} + i\alpha_a J_a + \frac{1}{2!}\frac{N(N-1)}{N^2}(i\alpha_a J_a)^2 + \frac{1}{3!}\frac{N(N-1)(N-2)}{N^3}(i\alpha_a J_a)^3 + \dots, \end{aligned} \tag{12}$$

where the Binomial theorem has been used. The transformation $g(\vec{\alpha})$ may become finite if we aggregate an infinite number of the infinitesimal transformations

$$g(\vec{\alpha}) = \lim_{N\to\infty} \left(\mathbf{I} + i\frac{\alpha_a}{N}J_a\right)^N = \mathbf{I} + i\alpha_a J_a + \frac{1}{2!}(i\alpha_a J_a)^2 + \frac{1}{3!}(i\alpha_a J_a)^3 + \dots \equiv \exp(i\alpha_a J_a), \tag{13}$$

where we have introduced the "exp" shorthand just to make life easier. Using the expansion (13) we can write

$$g(\vec{\alpha})g(\vec{\beta}) = e + i(\alpha_a + \beta_a)J_a - \frac{1}{2!}(\alpha_a\alpha_b + 2\alpha_a\beta_b + \beta_a\beta_b)J_aJ_b + \mathcal{O}(3), \tag{14}$$

where $\mathcal{O}(3)$ denotes terms proportional to $\alpha_a^p\beta_b^r$ with $p + r \geq 3$. As expected, it generally differs from $g(\vec{\alpha} + \vec{\beta})$ for which we get

$$g(\vec{\alpha} + \vec{\beta}) = e + i(\alpha_a + \beta_a)J_a - \frac{1}{2!}(\alpha_a\alpha_b + \alpha_a\beta_b + \beta_a\alpha_b + \beta_a\beta_b)J_aJ_b + \mathcal{O}(3). \tag{15}$$

Note that $(\alpha_a\beta_b + \beta_a\alpha_b)J_aJ_b = 2\alpha_a\beta_bJ_aJ_b - \alpha_a\beta_b[J_a, J_b]$.

[1] The generators can be formally understood as the first derivatives of the group elements at $\vec{\alpha} = \vec{0}$

$$iJ_a = \left.\frac{\partial g}{\partial \alpha_a}\right|_0 .$$

The local structure of a Lie group is tested when we compose the succession of four infinitesimal group elements $g(\vec{\varepsilon}_1)g(\vec{\varepsilon}_2)g^{-1}(\vec{\varepsilon}_1)g^{-1}(\vec{\varepsilon}_2)$. Obviously, if the elements commute the result is the unit element. In general however

$$g(\vec{\varepsilon}_1)g(\vec{\varepsilon}_2)g^{-1}(\vec{\varepsilon}_1)g^{-1}(\vec{\varepsilon}_2) = e - \varepsilon_{1a}\varepsilon_{2b}[J_a, J_b]. \tag{16}$$

The higher-order terms were neglected. So the local structure of any given Lie group is encoded in the value of the commutators of the group's generators. The role of the commutator gets confirmed by means of the *Baker-Campbell-Hausdorff relation* which we state here without a proof

$$e^A e^B = \exp\{A + B + \frac{1}{2!}[A, B] + \frac{1}{3!}(\frac{1}{2}[[A, B], B] + \frac{1}{2}[A, [A, B]]) + \ldots\}, \tag{17}$$

where $A$ and $B$ are finite vectors of the vector space of generators. Thus even the result of the composition of finite elements of a Lie group depends on the commutators $[J_a, J_b]$ whenever the finite elements are expressible in terms of the infinite expansion (13) at the unit element.

As we will demonstrate below all commutators $[J_a, J_b]$ can be expressed as linear combinations of the generators $J_a$. As a consequence the vector space of generators $J_a$ is closed with respect to the commutators. Thus the vector space of generators forms an algebra. The Taylor expansion of a group element is

$$g(\vec{\alpha}) = e + i\alpha_a J_a + \frac{1}{2!}\alpha_a\alpha_b J_{ab} + \ldots, \tag{18}$$

where the object $J_{ab} = J_{ba}$ have been introduced, formally equal to $(\partial^2 g/\partial\alpha_a\partial\alpha_b)_0$, and the dots indicate higher-order terms. The equation (18) implies

$$g(\vec{\alpha})g(\vec{\beta}) = e + i(\alpha_a + \beta_a)J_a - \alpha_a\beta_b J_a J_b + \frac{1}{2!}(\alpha_a\alpha_b + \beta_a\beta_b)J_{ab} + \mathcal{O}(3). \tag{19}$$

At the same time, using (9) in (18), we get

$$g(\vec{f}(\vec{\alpha}, \vec{\beta})) = e + i(\alpha_a + \beta_a)J_a + iC_{abc}\alpha_b\beta_c J_a + \frac{1}{2!}(\alpha_a\alpha_b + 2\alpha_a\beta_b + \beta_a\beta_b)J_{ab} + \mathcal{O}(3). \tag{20}$$

Comparing (19) and (20) we get

$$J_{ab} = -J_a J_b - iC_{cab}J_c. \tag{21}$$

The symmetry of $J_{ab}$ leads to

$$[J_a, J_b] = if_{abc}J_c, \tag{22}$$

where the objects

$$f_{abc} \equiv C_{cba} - C_{cab} = 2C_{cba} = -f_{bac} \tag{23}$$

are called the *group structure constants*. The eq. (21) rewritten in terms of $f_{abc}$ is

$$J_{ab} = -J_a J_b + \frac{i}{2}f_{abc}J_c. \tag{24}$$

The eq. (22) is the sought-after confirmation that the commutator of group generators can be expressed as a linear combination of the very same generators thus closing the vector space of the generators. Now, using (24) we can rewrite the Taylor expansion (18) of $R(\vec{\alpha})$ in the following way

$$R(\vec{\alpha}) = \mathbf{I} + i\alpha_a J_a + \frac{1}{2!}(i\alpha_a J_a)^2 + \mathcal{O}(\alpha^3), \tag{25}$$

This coincides with the power expansion of $R(\vec{\alpha})$ obtained in (13).

At the end we will derive a useful identity for the group structure constants. Applying the Jacobi identity, $[[A,B],C]+[[B,C],A]+[[C,A],B]=0$, to the generators $J_a$ and considering (22) we get $0=[[J_a,J_b],J_c]+[[J_b,J_c],J_a]+[[J_c,J_a],J_b]=-(f_{abd}f_{dce}+f_{bcd}f_{dae}+f_{cad}f_{dbe})J_e$. leading to

$$f_{abd}f_{dce}+f_{bcd}f_{dae}+f_{cad}f_{dbe}=0. \tag{26}$$

# 2 The $O(3)$ and $SO(3)$ Groups

Rotations in the 3-dim Euclidean space $E^{(3)}$ are represented by $3\times 3$ real regular matrices $R$

$$x\to x'=R(\vec{\alpha})x, \quad x,x'\in E^{(3)}. \tag{27}$$

The rotations are to maintain the norm $|x|=(x^T g x)^{1/2}$ — $g$ being the metric tensor — of the vector $x$

$$|x'|=|x|. \tag{28}$$

In the Euclidean space the metric is given by $g=\mathbf{I}$. The condition (28) applied to (27) leads to the *orthogonality* of matrices $R$

$$R^T R=\mathbf{I}. \tag{29}$$

This implies $\det R=\pm 1$.

The matrices for which the eq. (29) holds comprise the so-called $O(3)$ group. This restriction leads to a larger class of matrices than just those representing the rotations. In particular, the space reflection $R_P=-\mathbf{I}$ also fulfils the eq. (29) thus maintaining the length of a vector. Note that $\det R_P=-1$.

If the condition

$$\det R=1, \tag{30}$$

in addition to (29) is introduced the $R$ matrices form the so-called $SO(3)$ group. Its elements correspond to 3-dim rotations only. They are continuously connected to the unit element $R(\vec{0})=\mathbf{I}$ and can be expressed in an exponential form. An infinitesimal rotation has a form[2]

$$R(\vec{\varepsilon})=\mathbf{I}+i\Omega(\vec{\varepsilon}), \tag{31}$$

where $i\Omega$ is a real 3-dim matrix. When (31) is plugged into the eq. (29) we get

$$(i\Omega)^T=-(i\Omega). \tag{32}$$

It means that $i\Omega$ is an antisymmetric matrix and has three independent parameters. All $3\times 3$ antisymmetric matrices comprise a 3-dim real vector space. Therefore they can be expressed in a basis of three independent antisymmetric matrices $iJ_a$. A simple choice is

$$(iJ_a)_{bc}=\varepsilon_{abc}, \quad a,b,c=1,2,3. \tag{33}$$

The finite rotation can be written as

$$R(\vec{\alpha})=\exp(i\alpha_a J_a), \tag{34}$$

[2]Factor $i$ in front of $\Omega$ is a matter of convention. When introduced group generators become hermitian.

where matrices $R$ are the *fundamental representation* of the $SO(3)$ group. The generators $J_a$ are hermitian matrices with zeros on the main diagonal and imaginary numbers elsewhere. Their algebra is defined by

$$[J_a, J_b] = i\varepsilon_{abc}J_c \tag{35}$$

which can be verified using the explicit form (33).

Matrices for finite 3-dim rotations by an angle $\phi$ around individual cartesian axes possess the following form

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\phi & s_\phi \\ 0 & -s_\phi & c_\phi \end{pmatrix}, \quad R_y = \begin{pmatrix} c_\phi & 0 & -s_\phi \\ 0 & 1 & 0 \\ s_\phi & 0 & c_\phi \end{pmatrix}, \quad R_z = \begin{pmatrix} c_\phi & s_\phi & 0 \\ -s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{36}$$

where $s_\phi = \sin\phi$ and $c_\phi = \cos\phi$.

The *tensor representations* of $SO(3)$ can be build by means of multiple direct products of the fundamental (vector) representation. The tensor repesentations act on *tensors*. For example, a rank-2 tensor $T$ transforms as a direct product of two vectors

$$T' = (R \otimes R)T = RTR^T. \tag{37}$$

There is a rank-2 tensor invariant under $SO(3)$ transformations. It is a 3-dim unit matrix $\mathbf{I}^{(3)} = \mathrm{diag}(1, 1, 1)$. Its invariance is a direct consequence of the first group definition relation (29) which can be rewritten in the form

$$R\mathbf{I}^{(3)}R^T = \mathbf{I}^{(3)}. \tag{38}$$

The second group definition relation (30) implies the existence of a rank-3 invariant tensor: the 3-dim Levi-Civita tensor $\varepsilon^{(3)}$. The component form of the equation $\det R = 1$ reads $R_{1i}R_{2j}R_{3k}\varepsilon_{ijk} = 1$. Since $R_{1i}R_{1j}R_{nk}\varepsilon_{ijk} = R_{2i}R_{2j}R_{nk}\varepsilon_{ijk} = R_{3i}R_{3j}R_{nk}\varepsilon_{ijk} = 0$ we get $R_{\ell i}R_{mj}R_{nk}\varepsilon_{ijk} = \varepsilon_{\ell mn}$ which in compact form reads

$$(R \otimes R \otimes R)\varepsilon^{(3)} = \varepsilon^{(3)}. \tag{39}$$

# 3 The $SU(2)$ Group

The $SU(2)$ denotes a group of $2 \times 2$ complex matrices $U$ complying with

$$U^\dagger U = \mathbf{I}, \tag{40}$$

$$\det U = 1. \tag{41}$$

The $U$ matrix can be expressed in an exponential form

$$U(\vec{\alpha}) = \exp(i\Omega(\vec{\alpha})) = 1 + i\Omega(\vec{\alpha}) + \mathcal{O}(\alpha^2). \tag{42}$$

Substituting an infinitesimal $U$ into (40) we obtain

$$\Omega^\dagger = \Omega. \tag{43}$$

In addition, the condition (41) leads to $\mathrm{Tr}\,\Omega = 0$. Thus $\Omega$ is a Hermitian $2 \times 2$ complex matrix of a zero trace. The matrix of this sort has three independent real parameters and can be expressed in a basis of three independent Hermitian $2 \times 2$ complex matrices of a zero trace. The Pauli matrices

$\sigma_1, \sigma_2, \sigma_3$ would serve as the basis quite well. However, the usual choice of the $SU(2)$ generators is $\tau_i = \sigma_i/2$. Their algebra is given by

$$[\tau_a, \tau_b] = i\varepsilon_{abc}\tau_c, \tag{44}$$

which is identical to (35). Apparently, the $SU(2)$ generators obey the same algebra as the generators of the $SO(3)$ group. We can write the exponential parameterization of the $SU(2)$ group as

$$U(\vec{\alpha}) = \exp(i\vec{\alpha}\vec{\tau}), \qquad \vec{\tau} = \vec{\sigma}/2 \tag{45}$$

where $\vec{\alpha} = (\alpha^1, \alpha^2, \alpha^3)$ are real parameters. The identical algebras of $SU(2)$ and $SO(3)$ imply a local isomorphism of the groups in a vicinity of their unit elements.

The $U$ matrices transform vectors of a 2-dim complex vector space

$$\xi \rightarrow U\xi, \qquad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}. \tag{46}$$

At the same time matrices[3] $\bar{U} \equiv U^* = (U^\dagger)^T$ also comprise a representation of the $SU(2)$ group

$$\bar{\chi} \rightarrow \bar{U}\bar{\chi}, \qquad \bar{\chi} = \begin{pmatrix} \bar{\chi}_1 \\ \bar{\chi}_2 \end{pmatrix}. \tag{47}$$

We say that $\xi$ transform under the covariant representation and $\bar{\chi}$ under the contravariant one. Using vectors of both representations we can construct an invariant of the $SU(2)$ transformations

$$\bar{\chi}^T\xi = \bar{\chi}_i\xi_i \rightarrow \bar{\chi}^T U^\dagger U\xi = \bar{\chi}^T\xi. \tag{48}$$

In fact, the covariant and contravariant representations of $SU(2)$ are equivalent, i.e. there is a $2 \times 2$ regular matrix $\epsilon$ such that

$$\bar{U} = \epsilon^{-1}U\epsilon = \bar{\epsilon}U\epsilon. \tag{49}$$

where we have defined $\bar{\epsilon} \equiv \epsilon^{-1}$. Then (49) implies that $\bar{\epsilon}\xi$ transforms as $\bar{\xi}$ and $\epsilon\bar{\chi}$ transforms as $\chi$

$$(\bar{\epsilon}\xi') = (\bar{\epsilon}U\epsilon)(\bar{\epsilon}\xi) = \bar{U}(\bar{\epsilon}\xi), \tag{50}$$

$$(\epsilon\bar{\chi}') = (\epsilon\bar{U}\bar{\epsilon})(\epsilon\bar{\chi}) = U(\epsilon\bar{\chi}). \tag{51}$$

Considering (45) we can identify the matrix $\epsilon$ of (49) as

$$\epsilon = i\sigma^2 = \varepsilon^{(2)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \bar{\epsilon} = -i\sigma^2 = -\epsilon = \epsilon^T. \tag{52}$$

Indeed, since $\bar{\epsilon}\sigma_a\epsilon = -(\sigma_a)^*$, $a = 1, 2, 3$, we get

$$\bar{U}(\vec{\alpha}) = \bar{\epsilon}\exp(i\vec{\alpha}\vec{\tau})\epsilon = \exp(-i\vec{\alpha}\vec{\tau}^*) = U^*(\vec{\alpha}). \tag{53}$$

The Eq. (52) implies that

$$\bar{\epsilon}\xi = \begin{pmatrix} -\xi_2 \\ \xi_1 \end{pmatrix}, \qquad \epsilon\bar{\xi} = \begin{pmatrix} \xi_2^* \\ -\xi_1^* \end{pmatrix}. \tag{54}$$

[3]The bar over $U$ denotes a representation distinct from $U$. The asterisk means the complex conjugation of $U$.

We can build the so-called *tensor representations* of $SU(2)$ through the direct products of the covariant as well as contravariant representations. An important example is a rank-2 tensor $T$ which transforms as the direct product of covariant and contravariant vectors

$$\xi \otimes \bar{\chi} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} (\bar{\chi}_1, \bar{\chi}_2) = \begin{pmatrix} \xi_1\bar{\chi}_1 & \xi_1\bar{\chi}_2 \\ \xi_2\bar{\chi}_1 & \xi_2\bar{\chi}_2 \end{pmatrix} \equiv \xi\bar{\chi}^T, \tag{55}$$

or $(\xi \otimes \bar{\chi})_{ij} = \xi_i\bar{\chi}_j$. The $SU(2)$ transformation of the direct product is

$$\xi_i\bar{\chi}_j = (\xi\bar{\chi}^T)_{ij} \to U_{ik}\xi_k\bar{U}_{j\ell}\bar{\chi}_\ell = U_{ik}\xi_k\bar{\chi}_\ell U^\dagger_{\ell j} = (U\xi)_i(\bar{\chi}^T U^\dagger)_j = (U\xi\bar{\chi}^T U^\dagger)_{ij}, \tag{56}$$

or

$$\xi \otimes \bar{\chi} \to U(\xi \otimes \bar{\chi})U^\dagger. \tag{57}$$

Since $T$ transforms as the direct product, $T \sim \xi \otimes \bar{\chi}$, then

$$T \to UTU^\dagger. \tag{58}$$

Let us denote the representation transforming the tensor $T$ as $(1,1)$; the notation indicates the number of covariant and contravariant components in the direct product transformation equivalent.

The $(1,1)$ representation is not irreducible. Since $T$ has four independent complex components it belongs to a 4-dim complex vector space $V^{(4)}$ closed under $SU(2)$ transformations. However, the vector space has subspaces which themselves are closed under these transformations. First of all, following (58) we find that

$$\mathrm{Tr}(T') = \mathrm{Tr}(UTU^\dagger) = \mathrm{Tr}(T) \tag{59}$$

which is compatible with (48) since $\mathrm{Tr}(\xi \otimes \bar{\chi}) = (\xi \otimes \bar{\chi})_{ij}\delta_{ij} = \xi_i\bar{\chi}_j\delta_{ij} = \bar{\chi}^T\xi$. Thus the trace of $T$ forms a one-dimensional invariant subspace $V^{(1)} \subset V^{(4)}$. The remaining part of the $V^{(4)}$ space is populated with traceless[4] tensors $T - \mathbf{I}^{(2)}/2 \cdot \mathrm{Tr}(T)$. Let us mention that the transformation relation (58) along with (41) also leads to invariance of $\det T$

$$\det T' = \det(UTU^\dagger) = \det T. \tag{60}$$

In addition, the transformation (58) maintains hermiticity as well as antihermiticity of the tensors

$$T^\dagger = \pm T \qquad \Rightarrow \qquad (T')^\dagger = (UTU^\dagger)^\dagger = \pm T'. \tag{61}$$

Note that each matrix $M$ can be written as a sum of hermitian and antihermitian matrices. Indeed, if we define $H \equiv (M + M^\dagger)/2$ and $A \equiv (M - M^\dagger)/2$ then $H^\dagger = H$ and $A^\dagger = -A$ while $M = H + A$.

There was the homomorphism between the groups $SO(3)$ and $SU(2)$ mentioned above. To demonstrate it let us consider a traceless hermitian tensor $H$ transforming under $(1,1)$ representation of $SU(2)$

$$H(x) \equiv x_a\sigma_a = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}, \tag{62}$$

[4]'Traceless' means that the trace of the tensor is equal to zero.

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{63}$$

are the Pauli matrices. The determinant of $H$ is

$$\det H = -x_a x_a = -|x|^2. \tag{64}$$

Let $U$ be an $SU(2)$ matrix transforming $H(x)$ to

$$H(x') = UH(x)U^\dagger, \tag{65}$$

where $H(x') = x'_a\sigma_a$ and $H(x) = x_a\sigma_a$. Because of (60) the $SU(2)$ transformations of $H$ maintain the quantity $|x|$. This suggests a correspondence between $SU(2)$ and $SO(3)$ transformations. Since the $SU(2)$ transformation conserves $|x|$ there must be a matrix $R$ of $SO(3)$ transforming the Euclidean vector $x = (x_1, x_2, x_3)$ to the vector $x' = Rx = (x'_1, x'_2, x'_3)$. Thus

$$U(x_a\sigma_a)U^\dagger = (Rx)_a\sigma_a = R_{ab}x_b\sigma_a$$

Comparing factors at the corresponding coordinates we obtain

$$U\sigma_a U^\dagger = R_{ba}\sigma_b. \tag{66}$$

This relation provides an explicit mapping between the elements of the group $SU(2)$ and $SO(3)$. The mapping is a 2-1 homomorphism[5]: $\pm U \to R$. The eq. (66) also demonstrates that while the individual Pauli matrices transform as the $(1,1)$ representation of the group $SU(2)$ the triplet of the Pauli matrices form an $SO(3)$ vector. It means that the index $a$ on $\sigma_a$ is an $SO(3)$ vector index.

# 4 The $SU(2) \otimes SU(2)$ group

Let us introduce $L, R$ indices to distinguish the $SU(2)$ components of the $SU(2) \otimes SU(2)$ group. Then $SU(2)_L \otimes SU(2)_R$ is a group of the doublets $(g_L, g_R)$ where $g_{L,R} \in SU(2)_{L,R}$. The doublets follow the composition law

$$(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1h_2). \tag{67}$$

The six generators $J_a^{L,R}$ $(a = 1, 2, 3)$ of the group can be chosen to obey the following commutation relations

$$[J_a^{L,R}, J_b^{L,R}] = i\varepsilon_{abc}J_c^{L,R}, \quad [J_a^L, J_b^R] = 0. \tag{68}$$

The very obvious choice of matrices $J_a^{L,R}$ fulfilling the commutation relations (68) is the 4-dim matrices

$$J_a^R = \frac{1}{2}\begin{pmatrix} \sigma_a & 0 \\ 0 & 0 \end{pmatrix}, \quad J_a^L = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \sigma_a \end{pmatrix}. \tag{69}$$

[5]Note that $\det U = 1 \Rightarrow \det(-U) = 1$.

They can be expressed as the direct product[6] of the Pauli matrices and the projection matrices $P_{R,L} = (\mathbf{I}^{(2)} \pm \sigma_3)/2$

$$J_a^{R,L} = \frac{1}{2}\sigma_a \otimes P_{R,L}. \tag{70}$$

The exponential form of this 4-dim representation reads

$$U(\alpha_R, \alpha_L) = \exp[i(\alpha_a^R J_a^R + \alpha_b^L J_b^L)] = \exp(i\alpha_a^R J_a^R)\exp(i\alpha_b^L J_b^L) = U_R(\alpha_R)U_L(\alpha_L), \tag{71}$$

where matrices $U_R$ and $U_L$ commute. Explicitly

$$U_R(\alpha_R) = \begin{pmatrix} \exp(i\alpha_a^R\sigma_a/2) & 0 \\ 0 & \mathbf{I}^{(2)} \end{pmatrix}, \quad U_L(\alpha_L) = \begin{pmatrix} \mathbf{I}^{(2)} & 0 \\ 0 & \exp(i\alpha_a^L\sigma_a/2) \end{pmatrix}. \tag{72}$$

This representation transforms 4-dim complex vector $\xi$. The vector has two 2-dim parts, $\xi_r$ and $\xi_\ell$, each transforming as a $SU(2)$ spinor

$$\xi = \begin{pmatrix} \xi_r \\ \xi_\ell \end{pmatrix} \quad \to \quad U\xi = \begin{pmatrix} U_r\xi_r \\ U_\ell\xi_\ell \end{pmatrix}, \quad U_{r,\ell} = \exp(i\alpha_a^{R,L}\sigma_a/2). \tag{73}$$

Thus there are two 2-dim subspaces — the "left" one and the "right" one — not communicating with each other when experiencing $SU(2)_L \otimes SU(2)_R$ transformations. The 2-dim matrices $U_r$ and $U_\ell$ represent the $SU(2)_L \otimes SU(2)_R$ group in the right and left invariant subspaces, respectively.

We can built tensor representations of $SU(2)_L \otimes SU(2)_R$ as direct products of $U_r$ and $U_\ell$ matrices. In addition, we have learnt in Section 3 that besides the covariant representations $U_{r,\ell}$ there are also the contravariant representations $\bar{U}_{r,\ell}$. Hence, for example, a tensor which transforms as the direct product of the left and right spinors, $\xi_\ell \otimes \bar{\chi}_r$, is transformed by $U_\ell \otimes \bar{U}_r$ which can be written as

$$T \to U_\ell T U_r^\dagger, \quad T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{74}$$

where $a, b, c, d$ are complex numbers. The tensor $T$ belongs to a 4-dim complex vector space. The trace of $T^\dagger T$

$$\mathrm{Tr}(T^\dagger T) = |a|^2 + |b|^2 + |c|^2 + |d|^2 \tag{75}$$

is an $SU(2)_L \otimes SU(2)_R$ invariant object

$$\mathrm{Tr}(T^\dagger T) \to \mathrm{Tr}(U_R T^\dagger U_L^\dagger U_L T U_R^\dagger) = \mathrm{Tr}(T^\dagger T). \tag{76}$$

The $U_\ell \otimes \bar{U}_r$ representation is reducible. There is an $SU(2)_L \otimes SU(2)_R$ invariant 2-dim subspace of the vector space of $T$'s made up of matrices

$$M = \begin{pmatrix} d^* & b \\ -b^* & d \end{pmatrix} = (\tilde{\Phi}, \Phi) = \begin{pmatrix} \Upsilon^T \\ \tilde{\Upsilon}^T \end{pmatrix}, \tag{77}$$

[6] Throughout the notes we use the convention $A \otimes B = \begin{pmatrix} AB_{11} & AB_{12} \\ AB_{21} & AB_{22} \end{pmatrix}$ for the direct product of matrices. If $A \otimes B = \begin{pmatrix} A_{11}B & A_{12}B \\ A_{21}B & A_{22}B \end{pmatrix}$ were used then the order of the product factors in (70), and elsewhere, would get reversed.

where $b, d$ are complex numbers, and

$$\Phi \equiv \begin{pmatrix} b \\ d \end{pmatrix}, \quad \tilde{\Phi} \equiv \begin{pmatrix} d^* \\ -b^* \end{pmatrix} = \epsilon\Phi^*, \quad \Upsilon \equiv \begin{pmatrix} d^* \\ b \end{pmatrix}, \quad \tilde{\Upsilon} \equiv \begin{pmatrix} -b^* \\ d \end{pmatrix} = \bar{\epsilon}\Upsilon^*. \tag{78}$$

The tilded vectors transform in the same way as untilded ones: so if $\Phi \to U_\ell \Phi$ then $\tilde{\Phi} \to U_\ell \tilde{\Phi}$, and, if $\Upsilon \to \bar{U}_r \Upsilon$ then $\tilde{\Upsilon} \to \bar{U}_r \tilde{\Upsilon}$. Thus both, $U_\ell$ as well as $\bar{U}_r$, save the structure (77) of the matrix $M$

$$M = (\tilde{\Phi}, \Phi) \to U_\ell M = (U_\ell \tilde{\Phi}, U_\ell \Phi) = (\tilde{\Phi}', \Phi'), \tag{79}$$

where $\tilde{\Phi}' = \epsilon \Phi'^*$, and

$$M = \begin{pmatrix} \Upsilon^T \\ \tilde{\Upsilon}^T \end{pmatrix} \to M U_r^\dagger = \begin{pmatrix} (\bar{U}_r \Upsilon)^T \\ (\bar{U}_r \tilde{\Upsilon})^T \end{pmatrix} = \begin{pmatrix} \Upsilon'^T \\ \tilde{\Upsilon}'^T \end{pmatrix}, \tag{80}$$

where $\tilde{\Upsilon}' = \bar{\epsilon}\Upsilon'^*$.

There is a homomorphism between $SU(2)_L \otimes SU(2)_R$ and $SO(4)$. It can be seen if $M$ is parameterized in the following way

$$b = \frac{1}{\sqrt{2}}(x_2 + i x_1), \quad d = \frac{1}{\sqrt{2}}(x_4 - i x_3), \tag{81}$$

where $x_i$ are real numbers. Then

$$M = \frac{1}{\sqrt{2}} \begin{pmatrix} x_4 + i x_3 & x_2 + i x_1 \\ -x_2 + i x_1 & x_4 - i x_3 \end{pmatrix} = \frac{1}{\sqrt{2}}(\mathbf{I}^{(2)} x_4 + i x_a \sigma_a), \tag{82}$$

where $a = 1, 2, 3$. Using (75) we obtain

$$\mathrm{Tr}(M^\dagger M) = 2(|b|^2 + |d|^2) = x_a x_a, \tag{83}$$

where $a = 1, 2, 3, 4$. Due to (76) the object $x_a x_a$ is $SU(2)_L \otimes SU(2)_R$ invariant. The same object is invariant under $SO(4)$ transformations if $x_a$'s are components of an $SO(4)$ vector $(x_1, x_2, x_3, x_4)^T \equiv x^T$. For each $SU(2)_L \otimes SU(2)_R$ transformation $M(x') = U_L M(x) U_R^\dagger$ there is an $SO(4)$ rotation $x' = Rx$. Note that the $(-U_L, -\bar{U}_R)$ transformation induces the same $SO(4)$ rotation as $(U_L, \bar{U}_R)$.

We introduce another basis of the $SU(2)_L \otimes SU(2)_R$ algebra defining matrices

$$J_a = J_a^R + J_a^L, \quad K_a = J_a^R - J_a^L. \tag{84}$$

Their commutators are

$$[J_a, J_b] = i\varepsilon_{abc} J_c, \quad [K_a, K_b] = i\varepsilon_{abc} J_c, \quad [J_a, K_b] = i\varepsilon_{abc} K_c. \tag{85}$$

From (69) we get explicit form of the generators $J_a$, $K_a$

$$J_a = \frac{1}{2}\begin{pmatrix} \sigma_a & 0 \\ 0 & \sigma_a \end{pmatrix} = \frac{1}{2}\sigma_a \otimes \mathbf{I}^{(2)}, \quad K_a = \frac{1}{2}\begin{pmatrix} \sigma_a & 0 \\ 0 & -\sigma_a \end{pmatrix} = \frac{1}{2}\sigma_a \otimes \sigma_3. \tag{86}$$

The group elements generated by $J_a$'s form an $SU(2)$ subgroup of $SU(2)_L \otimes SU(2)_R$. The subgroup has a couple of names: *vector, diagonal*; usually it is denoted as $SU(2)_V$ or $SU(2)_{L+R}$.

The $G = SU(2)_L \otimes SU(2)_R$ group contains several subgroups. They are listed in the table below. There, a general element $g \in G$ is parameterized as $g(\vec{\alpha}_L, \vec{\alpha}_R) = \exp[i(\alpha_a^L J_a^L + \alpha_b^R J_b^R)]$, $a, b = 1, 2, 3$.

| subgroup $H$ | $g(\vec{\alpha}_L, \vec{\alpha}_R)$ | note |
|---|---|---|
| $SU(2)_L$ | $\vec{\alpha}_R = 0$ | |
| $SU(2)_R$ | $\vec{\alpha}_L = 0$ | |
| $SU(2)_V$ | $\vec{\alpha}_L = \vec{\alpha}_R$ | |
| $U(1)_{L3}$ | $\vec{\alpha}_L = (0, 0, \alpha_L), \vec{\alpha}_R = 0$ | $i)$ also $gHg^{-1}, g \in G$<br>$ii)$ $\subset SU(2)_L$ |
| $U(1)_{R3}$ | $\vec{\alpha}_L = 0, \vec{\alpha}_R = (0, 0, \alpha_R)$ | $i)$ also $gHg^{-1}, g \in G$<br>$ii)$ $\subset SU(2)_R$ |
| $U(1)_{L3} \otimes U(1)_{R3}$ | $\vec{\alpha}_L = (0, 0, \alpha_L), \vec{\alpha}_R = (0, 0, \alpha_R)$ | also<br>$g_1(U(1)_{L3})g_1^{-1} \otimes g_2(U(1)_{R3})g_2^{-1}$,<br>$g_1, g_2 \in G$ |
| $U(1)_{V3}$ | $\vec{\alpha}_L = (0, 0, \alpha), \vec{\alpha}_R = (0, 0, \alpha)$ | $i)$ also $gHg^{-1}, g \in SU(2)_V$<br>$ii)$ $\subset SU(2)_V$<br>$iii)$ $\subset U(1)_{L3} \otimes U(1)_{R3}$ |
| $U(1)_{A3}$ | $\vec{\alpha}_L = (0, 0, -\beta), \vec{\alpha}_R = (0, 0, \beta)$ | $i)$ also $gHg^{-1}, g \in SU(2)_V$<br>$ii)$ $\subset U(1)_{L3} \otimes U(1)_{R3}$ |
| $U(1)_{V3} \otimes U(1)_{A3}$ | $\vec{\alpha}_L = (0, 0, \alpha - \beta), \vec{\alpha}_R = (0, 0, \alpha + \beta)$ | $= U(1)_{L3} \otimes U(1)_{R3}$ |

(87)

The set of elements $g(\vec{\alpha}_L, \vec{\alpha}_R) \in G$ where $\vec{\alpha}_L = -\vec{\alpha}_R$ is usually denoted as $SU(2)_A$. It does not comprise a group, even though $SU(2)_L \otimes SU(2)_R = SU(2)_A \otimes SU(2)_V$. The direct product on the r.h.s. means that any $g \in G$ can be decomposed in the form $g = \xi h$ where $\xi \in SU(2)_A$, $h \in SU(2)_V$.

# 5 The $SO(4)$ group

$SO(4)$ is a group of 4-dim real orthogonal matrices $R$ with $\det R = 1$. The orthogonality condition is connected to the invariance of the length of the 4-dim real vector in the Euclidean space: $x^T x = (Rx)^T(Rx)$ is equvivalent to $R^T R = \mathbf{I}$. Thus $SO(4)$ is represented by the rotations in a 4-dim Euclidean space.

When the infinitesimal form $R = \mathbf{I} + i\Omega$ is plugged into the orthogonality condition it yields $(i\Omega)^T = -i\Omega$. Hence $\Omega$ is a real antisymmetric $4 \times 4$ matrix. The matrix is parameterizable with 6 real numbers. Thus $SO(4)$ group has 6 generators[7] $J_{ab} = -J_{ba}$, $(a, b = 1, 2, 3, 4)$ which must fulfill $J_{ab}^T = -J_{ab}$. They can be chosen as

$$(J_{ab})_{cd} = -i(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}). \tag{88}$$

As can be verified by an explicit calculation the algebra of the generators is given by[8]

$$[J_{ab}, J_{cd}] = i(\delta_{ac}J_{bd} - \delta_{ad}J_{bc} + \delta_{bd}J_{ac} - \delta_{bc}J_{ad}). \tag{89}$$

[7]The pair of indices on $J_{ab}$ serves to denote individual generators. It does not indicate an element of a matrix!

[8]Note that the $SO(3)$ algebra (35) can be obtained from (89) when we put $a, b, c = 1, 2, 3$, and $J_{ab} = \varepsilon_{abc}J_c$, which amounts to the substitution $J_{12} \to J_3$, $J_{23} \to J_1$, and $J_{31} \to J_2$.

Eventually, we can write down $R$ in the exponential form

$$R(\vec{\omega}) = \exp(i\omega_{ab}J_{ab}), \tag{90}$$

where $\omega_{ab} = -\omega_{ba}$ are six real parameters parameterizing the group $SO(4)$.

Let us rename the $J_{ab}$ generators in the following way

$$J_a \equiv \frac{1}{2}\varepsilon_{abc}J_{bc}, \quad K_a \equiv J_{a4}, \quad a,b,c = 1,2,3, \tag{91}$$

resulting in $J_1 = J_{23}$, $J_2 = J_{31}$, $J_3 = J_{12}$, $K_1 = J_{14}$, $K_2 = J_{24}$, and $K_3 = J_{34}$. Note that $J_{ab} = \varepsilon_{abc}J_c$ for $a,b,c = 1,2,3$. When we substitute (88) into (91) we obtain $(a,b,c = 1,2,3; j,k = 1,2,3,4)$

$$(J_a)_{jk} = \frac{1}{2}\varepsilon_{abc}(J_{bc})_{jk} = \begin{cases} -i\varepsilon_{ajk}, & j,k = 1,2,3 \\ 0, & j \vee k = 4 \end{cases} \tag{92}$$

The explicit form of the generators $J_a$, $K_a$ reads

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{93}$$

$$K_1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}. \tag{94}$$

We can verify that $J_1, J_2, J_3, K_1, K_2, K_3$ generate rotations in the $(2,3)$-, $(1,3)$-, $(1,2)$-, $(1,4)$-, $(2,4)$-, $(3,4)$- planes, respectively. For example, the infinitesimal transformation $R = \mathbf{I} + i\alpha J_1$ transforms coordinates of a vector $x = (x_1, x_2, x_3, x_4)$ as follows

$$x'_1 = x_1, \quad x'_2 = x_2 + \alpha x_3, \quad x'_3 = x_3 - \alpha x_2, \quad x'_4 = x_4. \tag{95}$$

This is obviously an infinitesimal rotation in the $(2,3)$-plane, because

$$(x'_2)^2 + (x'_3)^2 = (x_2 + \alpha x_3)^2 + (x_3 - \alpha x_2)^2 = (x_2)^2 + (x_3)^2$$

up to the $\mathcal{O}(\alpha^2)$. The infinitesimal transformation $R = \mathbf{I} + i\alpha K_1$ leads to

$$x'_1 = x_1 + \beta x_4, \quad x'_2 = x_2, \quad x'_3 = x_3, \quad x'_4 = x_4 - \beta x_1. \tag{96}$$

It represents an infinitesimal rotation in the $(1,4)$-plane, since $(x'_1)^2 + (x'_4)^2 = (x_1)^2 + (x_4)^2 + \mathcal{O}(\beta^2)$. The commutators of $J_a$'s and $K_a$'s can be derived from (89)

$$[J_a, J_b] = i\varepsilon_{abc}J_c, \quad [K_a, K_b] = i\varepsilon_{abc}J_c, \quad [J_a, K_b] = i\varepsilon_{abc}K_c. \tag{97}$$

We introduce yet another six new generators following the pattern $(J_{ab} \pm \varepsilon_{abcd}J_{cd})/2$

$$J_1^{\pm} = \frac{1}{2}(J_{23} \pm J_{14}), \quad J_2^{\pm} = \frac{1}{2}(J_{31} \pm J_{24}), \quad J_3^{\pm} = \frac{1}{2}(J_{12} \pm J_{34}). \tag{98}$$

The generators fulfill the following commutation relations

$$[J_a^\pm, J_b^\pm] = i\varepsilon_{abc} J_c^\pm, \quad [J_a^+, J_b^-] = 0. \tag{99}$$

The $J_a^\pm$ generators in the fundamental representation can be derived from (88) and expressed in terms of the direct product of the Pauli matrices and the 2-dim unit matrix

$$J_1^+ = \frac{1}{2}(\sigma_1 \otimes \sigma_2), \quad J_2^+ = -\frac{1}{2}(\sigma_3 \otimes \sigma_2), \quad J_3^+ = \frac{1}{2}(\sigma_2 \otimes \mathbf{I}^{(2)}), \tag{100}$$

$$J_1^- = -\frac{1}{2}(\sigma_2 \otimes \sigma_1), \quad J_2^- = -\frac{1}{2}(\mathbf{I}^{(2)} \otimes \sigma_2), \quad J_3^- = \frac{1}{2}(\sigma_2 \otimes \sigma_3). \tag{101}$$

The commutators (99) imply that the group $SO(4)$ is locally isomorphic to the group $SO(3) \otimes SO(3)$. However, recalling that $SO(3)$ is locally isomorphic to $SU(2)$ we conclude that $SO(4)$ is also locally isomorphic to $SU(2) \otimes SU(2)$.

The relation between the generators $J_a, K_a$ and $J_a^\pm$ is

$$J_a^\pm = \frac{1}{2}(J_a \pm K_a) \tag{102}$$

and inversely

$$J_a = J_a^+ + J_a^-, \quad K_a = J_a^+ - J_a^-. \tag{103}$$

In the fundamental representation, combining (100), (101), and (103), we get

$$J_1 = \frac{1}{2}(\sigma_1 \otimes \sigma_2 - \sigma_2 \otimes \sigma_1), \quad J_2 = -P_R \otimes \sigma_2, \quad J_3 = \sigma_2 \otimes P_R, \tag{104}$$

$$K_1 = \frac{1}{2}(\sigma_1 \otimes \sigma_2 + \sigma_2 \otimes \sigma_1), \quad K_2 = P_L \otimes \sigma_2, \quad K_3 = \sigma_2 \otimes P_L, \tag{105}$$

where $P_{R,L}^{(2)} = (\mathbf{I}^{(2)} \pm \sigma_3)/2$ are projection matrices in a 2-dim vector space.

# 6 The spinor representation of $SO(4)$

If we consider the commutation relations (89) as the defining property of the $SO(4)$ group then besides the vector and tensor representations considered in Section 5 we can construct the so-called *spinor representation* of the group. Let us introduce four 4-dim matrices $\Gamma_1, \ldots, \Gamma_4$ for which the following anticommutation relation holds

$$\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}\mathbf{I}^{(4)}. \tag{106}$$

The vector space of the gamma-matrices with the "product" (106) is called the *Clifford algebra.* The gamma-matrices can be used to build the generators of the $SO(4)$ spinor representation in the following way

$$\Sigma_{ab} = -\frac{i}{4}[\Gamma_a, \Gamma_b], \tag{107}$$

It can be verified that the $\Sigma$'s satisfy the $SO(4)$ algebra (89)

$$[\Sigma_{ab}, \Sigma_{cd}] = i(\delta_{ac}\Sigma_{bd} - \delta_{ad}\Sigma_{bc} + \delta_{bd}\Sigma_{ac} - \delta_{bc}\Sigma_{ad}). \tag{108}$$

The spinor representation is four-dimensional. It transforms complex vectors called *spinors*

$$\xi \to S\xi, \qquad S = \exp(i\alpha_{ab}\Sigma_{ab}). \tag{109}$$

If the gamma-matrices are (anti)hermitian the generators $\Sigma$'s are hermitians with zero trace. Then the transformation (109) is unitary, $S^\dagger S = \mathbf{I}$. In particular, when the gamma-matrices are real and (anti)symmetric, the matrices $i\Sigma_{ab}$ are real and antisymmetric. In this case $S$ is orthogonal, $S^T S = \mathbf{I}$. Real spinor representations are called *Majorana*'s.

The spinor representation of $SO(4)$ is reducible. To demonstrate it let us define the fifth gamma-matrix

$$\Gamma_5 \equiv \pm\Gamma_1\Gamma_2\Gamma_3\Gamma_4, \tag{110}$$

where the choice of the sign is arbitrary and does not affect any conclusions below. First of all, $\Gamma_5$ anticommutes with all original gamma-matrices

$$\{\Gamma_5, \Gamma_a\} = 0, \qquad a = 1, \ldots, 4. \tag{111}$$

Secondly[9],

$$\Gamma_5^2 = \mathbf{I}^{(4)}. \tag{112}$$

We can define

$$P_{R,L} = \frac{1}{2}(\mathbf{I}^{(4)} \pm \Gamma_5) \tag{113}$$

which are projection matrices because

$$P_R + P_L = \mathbf{I}^{(4)}, \qquad P_{R,L}^2 = P_{R,L}, \qquad P_R P_L = 0. \tag{114}$$

Let us list here some useful identities

$$P_R - P_L = \Gamma_5, \qquad P_{R,L}\Gamma_a = \Gamma_a P_{L,R}, \qquad P_{R,L}\Gamma_5 = \pm P_{R,L}, \tag{115}$$

$$[P_{R,L}, \Gamma_a] = \pm\Gamma_5\Gamma_a, \qquad [P_{R,L}, \Gamma_5] = 0. \tag{116}$$

The projection matrices split the spinor space into two distinct subspaces: the "right" spinors $\xi_R$ and the "left" spinors $\xi_L$

$$\xi_{R,L} \equiv P_{R,L}\xi, \qquad \xi_R + \xi_L = \xi. \tag{117}$$

The first equation of (116) implies that the projection matrices $P_{R,L}$ commute with all $SO(4)$ generators and thus with all $SO(4)$ transformations

$$[P_{R,L}, \Sigma_{ab}] = 0 \qquad \Rightarrow \qquad [P_{R,L}, S] = 0. \tag{118}$$

Then

$$S\xi_{R,L} = SP_{R,L}\xi = P_{R,L}S\xi = P_{R,L}\xi' = \xi'_{R,L}. \tag{119}$$

It means that the right and left subspaces are invariant under $SO(4)$ transformations.

The right and left spinors are eigenvectors of $\Gamma_5$ with eigenvalues $\pm 1$

$$\Gamma_5\xi_{R,L} = \Gamma_5 P_{R,L}\xi = \pm P_{R,L}\xi = \pm\xi_{R,L}. \tag{120}$$

[9]Note that all matrices of the set $\Gamma_1, \ldots, \Gamma_5$ fulfill the anticommutation relation (106). As a matter of fact these are generators of the $SO(5)$ spinor representation which is four-dimensional.

The representation in which $\Gamma_5$ is diagonal is called *chiral*. In this representation the eigenvalues of $\Gamma_5$ are spread along its main diagonal. Thus

$$\Gamma_5 = \begin{pmatrix} \mathbf{I}^{(n_R)} & 0 \\ 0 & -\mathbf{I}^{(n_L)} \end{pmatrix}, \qquad n_R + n_L = 4, \tag{121}$$

where we have grouped together the eigenvalues $+1$ and the eigenvalues $-1$. In this representation spinors have a form

$$\xi = \begin{pmatrix} \xi_r \\ \xi_\ell \end{pmatrix}, \qquad \xi_R = \begin{pmatrix} \xi_r \\ 0 \end{pmatrix}, \qquad \xi_L = \begin{pmatrix} 0 \\ \xi_\ell \end{pmatrix}, \tag{122}$$

where $\xi_{r,\ell}$ are called *Weyl spinors*. The left and right subspaces are both two-dimensional, $n_R = n_L = 2$. To prove it note that the eq. (106) implies $\mathrm{Tr}(\Gamma_1\Gamma_2\Gamma_3\Gamma_4) = -\mathrm{Tr}(\Gamma_4\Gamma_1\Gamma_2\Gamma_3)$. At the same time, due to general properties of the trace, $\mathrm{Tr}(\Gamma_1\Gamma_2\Gamma_3\Gamma_4) = \mathrm{Tr}(\Gamma_4\Gamma_1\Gamma_2\Gamma_3)$ holds. This results in $\mathrm{Tr}(\Gamma_5) = 0$. Since $\mathrm{Tr}(\Gamma_5) = n_R - n_L$ we get $n_R = n_L$. Since $n_R + n_L = 4$ we obtain $n_R = n_L = 2$. Thus $\Gamma_5$ can be chosen in the following way

$$\Gamma_5 = \begin{pmatrix} \mathbf{I}^{(2)} & 0 \\ 0 & -\mathbf{I}^{(2)} \end{pmatrix} = \mathbf{I}^{(2)} \otimes \sigma_3. \tag{123}$$

The projection matrices in the chiral representation assume forms

$$P_{R,L} = \frac{1}{2}(\mathbf{I}^{(4)} \pm \mathbf{I}^{(2)} \otimes \sigma_3) = \frac{1}{2}\mathbf{I}^{(2)} \otimes (\mathbf{I}^{(2)} \pm \sigma_3) = \begin{cases} \mathrm{diag}(\mathbf{I}^{(2)}, 0) \\ \mathrm{diag}(0, \mathbf{I}^{(2)}) \end{cases}. \tag{124}$$

There is also the complex conjugated spinor representation $\bar{S}$ of $SO(4)$

$$\bar{\chi} \to \bar{S}\bar{\chi}, \qquad \bar{S} = S^* = \exp(-i\alpha_{ab}\Sigma^*_{ab}). \tag{125}$$

In the case $\bar{S}$ is a unitary[10] representation it is identical to the contragradient representation $(S^{-1})^T$

$$S^* = (e^{i\alpha_{ab}\Sigma_{ab}})^* = e^{-i\alpha_{ab}(\Sigma_{ab})^T} = (S^{-1})^T. \tag{126}$$

As in Section 5 we can introduce the following set of generators

$$J_a \equiv \frac{1}{2}\varepsilon_{abc}\Sigma_{bc}, \qquad K_a \equiv \Sigma_{a4}, \qquad a,b,c = 1,2,3 \tag{127}$$

with the commutators

$$[J_a, J_b] = i\varepsilon_{abc}J_c, \qquad [K_a, K_b] = i\varepsilon_{abc}J_c, \qquad [J_a, K_b] = i\varepsilon_{abc}K_c. \tag{128}$$

And also the set

$$J^\pm_a \equiv (J_a \pm K_a) \tag{129}$$

with the commutators

$$[J^\pm_a, J^\pm_b] = i\varepsilon_{abc}J^\pm_c, \qquad [J^+_a, J^-_b] = 0. \tag{130}$$

[10] All representations of finite groups as well as all finite-dimensional representations of connected simple compact infinite groups are equivalent to a unitary representation.

The choice of the $\Gamma$-matrices satisfying the eq. (106) is not unique. If $\mathcal{S}$ is a regular 4-dim matrix then $\Gamma'_a = \mathcal{S}\Gamma_a\mathcal{S}^{-1}$ also satisfies the anticommutation relation. A convenient way to construct the gamma-matrices is through the direct product of the Pauli matrices and a 2-dim unit matrix. An example is

$$\Gamma_a = \sigma_a \otimes \sigma_1 = \begin{pmatrix} 0 & \sigma_a \\ \sigma_a & 0 \end{pmatrix}, \quad \Gamma_4 = \mathbf{I}^{(2)} \otimes \sigma_2 = \begin{pmatrix} 0 & -i\mathbf{I}^{(2)} \\ i\mathbf{I}^{(2)} & 0 \end{pmatrix}, \quad a = 1,2,3. \tag{131}$$

It results in a diagonal $\Gamma_5$ matrix

$$\Gamma_5 = \pm\Gamma_1\Gamma_2\Gamma_3\Gamma_4 = \pm(-\mathbf{I}^{(2)} \otimes \sigma_3) = \pm \begin{pmatrix} -\mathbf{I}^{(2)} & 0 \\ 0 & \mathbf{I}^{(2)} \end{pmatrix}, \tag{132}$$

which indicates that we deal with the chiral representation. The lower sign definition corresponds to the construction (123). The $\Sigma$-generators are

$$\Sigma_{ab} = -\frac{i}{4}[\Gamma_a, \Gamma_b] = \frac{1}{2}\varepsilon_{abc}(\sigma_c \otimes \mathbf{I}^{(2)}) = \frac{1}{2}\varepsilon_{abc} \begin{pmatrix} \sigma_c & 0 \\ 0 & \sigma_c \end{pmatrix}, \quad a,b,c = 1,2,3, \tag{133}$$

$$\Sigma_{a4} = -\frac{i}{4}[\Gamma_a, \Gamma_4] = \frac{1}{2}\sigma_a \otimes \sigma_3 = \frac{1}{2} \begin{pmatrix} \sigma_a & 0 \\ 0 & -\sigma_a \end{pmatrix}, \quad a = 1,2,3. \tag{134}$$

In the usual way they can be renamed as $J$'s and $K$'s

$$J_a = \frac{1}{2}\varepsilon_{abc}\Sigma_{bc} = \frac{1}{2}(\sigma_a \otimes \mathbf{I}^{(2)}) = \frac{1}{2} \begin{pmatrix} \sigma_a & 0 \\ 0 & \sigma_a \end{pmatrix}, \tag{135}$$

$$K_a = \Sigma_{a4} = \frac{1}{2}\sigma_a \otimes \sigma_3 = \frac{1}{2} \begin{pmatrix} \sigma_a & 0 \\ 0 & -\sigma_a \end{pmatrix}, \tag{136}$$

where $a,b,c = 1,2,3$. Consequently,

$$J_a^+ = (J_a + K_a) = \begin{pmatrix} \sigma_a & 0 \\ 0 & 0 \end{pmatrix}, \quad J_a^- = (J_a - K_a) = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_a \end{pmatrix}, \tag{137}$$

where $a = 1,2,3$.

In the case of the chiral representation a more compact notation can be introduced. Let us define

$$\sigma_a \equiv (\sigma_1, \sigma_2, \sigma_3, -i\mathbf{I}^{(2)}), \quad \bar{\sigma}_a \equiv (\sigma_1, \sigma_2, \sigma_3, i\mathbf{I}^{(2)}), \tag{138}$$

and

$$\sigma_{ab} \equiv -\frac{i}{4}(\sigma_a\bar{\sigma}_b - \sigma_b\bar{\sigma}_a), \quad \bar{\sigma}_{ab} \equiv -\frac{i}{4}(\bar{\sigma}_a\sigma_b - \bar{\sigma}_b\sigma_a). \tag{139}$$

Then

$$\Gamma_a = \begin{pmatrix} 0 & \sigma_a \\ \bar{\sigma}_a & 0 \end{pmatrix}, \quad \Sigma_{ab} = \begin{pmatrix} \sigma_{ab} & 0 \\ 0 & \bar{\sigma}_{ab} \end{pmatrix}, \tag{140}$$

where $a,b = 1,2,3,4$.

Suitable gamma-matrices can be obtained when $\sigma_1$ and $\sigma_2$ at the second position of the direct products in (131) are replaced by any pair of (mutually different) Pauli matrices. For example,

$$\Gamma_a = \sigma_a \otimes \sigma_2 = \begin{pmatrix} 0 & -i\sigma_a \\ i\sigma_a & 0 \end{pmatrix}, \quad \Gamma_4 = \mathbf{I}^{(2)} \otimes \sigma_3 = \begin{pmatrix} i\mathbf{I}^{(2)} & 0 \\ 0 & -i\mathbf{I}^{(2)} \end{pmatrix}, \quad a = 1, 2, 3. \tag{141}$$

It results in

$$\Gamma_5 = \pm\Gamma_1\Gamma_2\Gamma_3\Gamma_4 = \pm(-\mathbf{I}^{(2)} \otimes \sigma_1) = \pm \begin{pmatrix} 0 & -\mathbf{I}^{(2)} \\ -\mathbf{I}^{(2)} & 0 \end{pmatrix}. \tag{142}$$

The $\Sigma$-generators are

$$\Sigma_{ab} = -\frac{i}{4}[\Gamma_a, \Gamma_b] = \frac{1}{2}\varepsilon_{abc}(\sigma_c \otimes \mathbf{I}^{(2)}) = \frac{1}{2}\varepsilon_{abc} \begin{pmatrix} \sigma_c & 0 \\ 0 & \sigma_c \end{pmatrix}, \quad a, b, c = 1, 2, 3, \tag{143}$$

$$\Sigma_{a4} = -\frac{i}{4}[\Gamma_a, \Gamma_4] = \frac{1}{2}\sigma_a \otimes \sigma_1 = \frac{1}{2} \begin{pmatrix} 0 & \sigma_a \\ \sigma_a & 0 \end{pmatrix}, \quad a = 1, 2, 3. \tag{144}$$

The $J$'s and $K$'s are

$$J_a = \frac{1}{2}\varepsilon_{abc}\Sigma_{bc} = \frac{1}{2}(\sigma_a \otimes \mathbf{I}^{(2)}) = \frac{1}{2} \begin{pmatrix} \sigma_a & 0 \\ 0 & \sigma_a \end{pmatrix}, \tag{145}$$

$$K_a = \Sigma_{a4} = \frac{1}{2}\sigma_a \otimes \sigma_1 = \frac{1}{2} \begin{pmatrix} 0 & \sigma_a \\ \sigma_a & 0 \end{pmatrix}, \tag{146}$$

where $a, b, c = 1, 2, 3$. Consequently,

$$J_a^+ = (J_a + K_a) = \frac{1}{2} \begin{pmatrix} \sigma_a & \sigma_a \\ \sigma_a & \sigma_a \end{pmatrix}, \quad J_a^- = (J_a - K_a) = \frac{1}{2} \begin{pmatrix} \sigma_a & -\sigma_a \\ -\sigma_a & \sigma_a \end{pmatrix}, \tag{147}$$

where $a = 1, 2, 3$.

# 7 The $SO(1,3)$ group and the Lorentz transformations

The *Minkowski space* is a 4-dim vector space with the *pseudo*metric[11] given by the metric tensor

$$g \equiv \mathrm{diag}(1, -1, -1, -1). \tag{148}$$

Note that the inverse matrix of $g$ is the matrix $g$ itself, $g^{-1} = g$. Reflecting the pattern of the main diagonal of $g$ we will denote the Minkowski space as $V^{(1,3)}$. The squared "length" of a vector $x \in V^{(1,3)}$ is given as a *pseudo*scalar product of $x$ with itself

$$x^2 \equiv x^T g x = x_i g_{ij} x_j. \tag{149}$$

It can be seen easily that $x^2$ of a non-trivial $x$ is not positively definite as it would have been in an Euclidean space. It can assume zero or negative values as well. There is a convention

[11] The "metric" defined by the tensor (148) does not fulfill all the metric axioms. Namely, the requirement of the positivity of the norm of a vector.

that the index of Minkowski four-vector components runs over values $0, 1, 2, 3$. Thus $g_{00} = 1$, $g_{11} = g_{22} = g_{33} = -1$.

The Lorentz transformations (LT) of $x \in V^{(1,3)}$ maintain $x^2$. In this sense they can be understood as a Minkowski space analogue of rotations. Let $\Lambda$ be a 4-dim real matrix representing LT of a four-vector $x$

$$x \to x' = \Lambda x. \tag{150}$$

Then

$$x'^2 = x'^T g x' = x^T (\Lambda^T g \Lambda) x. \tag{151}$$

From $x'^2 = x^2$ we obtain the condition for $\Lambda$

$$\Lambda^T g \Lambda = g. \tag{152}$$

It is analogous to the orthogonality condition (29). It can be verified that LT's comprise a Lie group. Let us denote it as $O(1, 3)$.

The prototype of the Minkowski space is a space of four-vectors $x = (x_0, x_1, x_2, x_3)^T = (ct, \vec{x})^T$ where $t$ is a time coordinate, $c$ is a speed of light, and $\vec{x} = (x_1, x_2, x_3)$ is a position vector expressed in Cartesian coordinates. It is called the *spacetime*. The quantity invariant under LT in the spacetime is called the *interval* $s$ where

$$s^2 \equiv x^2 = (ct)^2 - \vec{x}^2. \tag{153}$$

If $s^2 > 0$ the four-vector $x$ is called *time-like*, if $s^2 < 0$ the four-vector is *space-like*, and, finally, for $s^2 = 0$ it is called *light-like*.

The eq. (152) implies

$$\det \Lambda = \pm 1, \tag{154}$$

and[12]

$$|\Lambda_{00}| \geq 1. \tag{155}$$

The LT's with $\det \Lambda = 1$ are called *proper* and denoted $\Lambda_+$. The product of two proper LT's is a proper LT. The identity transformation is a proper one. There is an inverse transformation to each proper LT and it is also a proper transformation. The Lorentz transformations $\Lambda_-$ with the determinant equal to $-1$ are said to be *improper*. The LT's where $\Lambda_{00} \geq 1$ are called *ortochronous* and denoted as $\Lambda^\uparrow$. The product of two ortochronous transformations is an ortochronous one. It includes the identity transformation. When $\Lambda_{00} \leq -1$ the LT's are said to be *non-ortochronous* and denoted $\Lambda^\downarrow$.

The LT's of the form

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}, \qquad R^T R = \mathbf{I}, \qquad \det R = 1, \tag{156}$$

where $R$ is a $3 \times 3$ real matrix represent rotations in a 3-dim Euclidean space. Not suprisingly they comprise an $O(3)$ subgroup of the Lorentz group $O(1, 3)$. In this case $\det \Lambda = 1$ and $\Lambda_{00} = 1$. Thus the rotations qualify for the proper ortochronous LT's.

[12] The 00-component of the eq. (152) reads $\Lambda_{00}^2 - \sum_{i=1}^{3} \Lambda_{i0}^2 = 1$. Since $\Lambda$'s are real matrices the equation results in (155).

Boosts between two inertial frames also belong to the proper ortochronous LT's. For example, the boost along the 1-direction is given by the matrix

$$\Lambda = \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \cosh\eta = \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}, \qquad \sinh\eta = \beta\gamma, \qquad \beta \equiv \frac{v}{c}, \tag{157}$$

where $v$ is a relative velocity of the frames, $v < c$, and $-\infty < \eta < \infty$. Since $\det\Lambda = \cosh^2\eta - \sinh^2\eta = 1$ and $\Lambda_{00} = \cosh\eta \geq 1$ the 1-boost is a proper and ortochronous LT. However, pure boosts do not comprise a group.

The proper ortochronous LT's $\Lambda_+^\uparrow$ comprise the so-called $SO(1,3)$ subgroup of the $O(1,3)$ group. Any $\Lambda_+^\uparrow$ transformation can be written as a product of a space rotation and a Lorentz boost. Generally, when a physical theory is said to be Lorentz invariant, the invariance under the proper ortochronous LT's is meant.

The squared spacetime interval $s^2$ also remains invariant under various spacetime inversions. Thus they also qualify for LT's. For example, the *space inversion* is represented by the matrix

$$\Lambda = \mathrm{diag}(1,-1,-1,-1) = g \equiv \Lambda_P. \tag{158}$$

Since $\det\Lambda = -1$ and $\Lambda_{00} = 1$ it is an <u>improper ortochronous</u> LT. The *time inversion* is represented by the matrix

$$\Lambda = \mathrm{diag}(-1,1,1,1) \equiv \Lambda_T. \tag{159}$$

Since $\det\Lambda = -1$ and $\Lambda_{00} = -1$ it is an <u>improper non-ortochronous</u> LT. Finally, the *full inversion*

$$\Lambda = \mathrm{diag}(-1,-1,-1,-1) = \Lambda_P\Lambda_T \tag{160}$$

is a <u>proper non-ortochronous</u> LT: $\det\Lambda = 1$ and $\Lambda_{00} = -1$. Any LT can be decomposed as the product of $\Lambda_+^\uparrow$, $\Lambda_P$, and $\Lambda_T$. The claim is based on the fact that the four classes of LT's, $\Lambda_+^\uparrow$, $\Lambda_-^\uparrow$, $\Lambda_+^\downarrow$, $\Lambda_-^\downarrow$, cover all LT's, and on the observation that

$$\Lambda_-^\uparrow\Lambda_P,\ \Lambda_-^\downarrow\Lambda_T,\ \Lambda_+^\downarrow\Lambda_P\Lambda_T \in \{\Lambda_+^\uparrow\}. \tag{161}$$

Since all the inversion matrices are identical to their inverse matrices, $\Lambda_{P,T,PT}^2 = \mathbf{I}$, then for every $\Lambda$ there is $\Lambda_+^\uparrow$ such that $\Lambda = \Lambda_+^\uparrow\Lambda_{P,T,PT}$.

The proper ortochronous Lorentz transformations[13] can be expressed in the exponential form. When we plug an infinitesimal LT, $\Lambda = \mathbf{I} + i\Omega$, into the condition (152) we get

$$(i\Omega)^T = -g(i\Omega)g, \tag{162}$$

where we took into account $g^{-1} = g$. To find out what kind of matrix $\Omega$ fits the condition (162) we can rewrite it as

$$[g(i\Omega)]^T = -g(i\Omega). \tag{163}$$

This is exactly the condition the generators of the $SO(4)$ group have to fulfill. Thus $g(i\Omega)$ is a real antisymmetric $4\times 4$ matrix. As in the $SO(4)$ case it has 6 independent real parameters.

[13] Henceforth, unless specified otherwise, the proper ortochronous LT's will be refered to as the Lorentz transformations.

Consequently, the Lorentz group $SO(1,3)$ also has six generators. Following (88) the $SO(1,3)$ generators $J_{ab} = -J_{ba}$ $(a, b = 0, 1, 2, 3)$ can be chosen to comprise with the relation

$$(gJ_{ab})_{cd} = -i(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}), \tag{164}$$

or

$$(J_{ab})_{cd} = -i(g_{ac}\delta_{bd} - \delta_{ad}g_{bc}). \tag{165}$$

Obviously, the commutation relation (89) must hold for $(gJ_{ab})$ matrices. From there, or by explicit calculation, we can verify that

$$[J_{ab}, J_{cd}] = i(g_{ac}J_{bd} - g_{ad}J_{bc} + g_{bd}J_{ac} - g_{bc}J_{ad}). \tag{166}$$

Then the exponential form of the LT is

$$\Lambda(\vec{\omega}) = \exp(i\omega_{ab}J_{ab}), \qquad a, b = 0, 1, 2, 3, \tag{167}$$

where $\omega_{ab} = -\omega_{ba}$ are six real parameters.

Let us define a new set of $SO(1,3)$ generators[14]

$$J_a \equiv -\frac{1}{2}\varepsilon_{abc}J_{bc}, \qquad K_a \equiv J_{a0}, \qquad a, b, c = 1, 2, 3, \tag{168}$$

resulting in $J_1 = -J_{23}$, $J_2 = -J_{31}$, $J_3 = -J_{12}$, $K_1 = J_{10}$, $K_2 = J_{20}$, and $K_3 = J_{30}$. The inverse relation to the definition of $J_a$ reads $J_{ab} = -\varepsilon_{abc}J_c$. We can see that (168) is, up to the sign, more like renaming the generators $J_{ab}$. When we substitute (165) into (168) we obtain $(a, b, c = 1, 2, 3; j, k = 0, 1, 2, 3)$

$$(J_a)_{jk} = -\frac{1}{2}\varepsilon_{abc}(J_{bc})_{jk} = \begin{cases} -i\varepsilon_{ajk}, & j, k = 1, 2, 3 \\ 0, & j \vee k = 0 \end{cases} \tag{169}$$

Then the explicit form of the generators $J_a$, $K_a$ reads

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{170}$$

$$K_1 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}. \tag{171}$$

The matrices $J_1, J_2, J_3$ generate 3-dim rotations around the axes $1, 2, 3$, respectively. The matrices $K_1, K_2, K_3$ generate boosts along the axes $1, 2, 3$, respectively. For example, the infinitesimal transformation $\Lambda = \mathbf{I} + i\varphi J_1$ transforms the four-vector $x = (x_0, x_1, x_2, x_3)$ as follows

$$x'_0 = x_0, \qquad x'_1 = x_1, \qquad x'_2 = x_2 + \varphi x_3, \qquad x'_3 = x_3 - \varphi x_2, \tag{172}$$

[14] The choice $J_a \equiv -\frac{1}{2}\varepsilon_{abc}J_{bc}$, $K_a \equiv -J_{a0}$ would work as well.

This is obviously an infinitesimal rotation in the $(2,3)$-plane, because

$$(x_2')^2 + (x_3')^2 = (x_2 + \varphi x_3)^2 + (x_3 - \varphi x_2)^2 = (x_2)^2 + (x_3)^2$$

up to the $\mathcal{O}(\varphi^2)$. The infinitesimal transformation $\Lambda = \mathbf{I} + i\eta K_1$ leads to

$$x_0' = x_0 - \eta x_1, \quad x_1' = x_1 - \eta x_0, \quad x_2' = x_2, \quad x_3' = x_3. \tag{173}$$

This is an infinitesimal boost, since

$$(x_0')^2 - (x_1')^2 = (x_0 - \eta x_1)^2 - (x_1 - \eta x_0)^2 = (x_0')^2 - (x_1')^2 + \mathcal{O}(\eta^2).$$

The commutation relations for the generators $J_a$, $K_a$ can be obtained from (166)

$$[J_a, J_b] = i\varepsilon_{abc} J_c, \quad [K_a, K_b] = -i\varepsilon_{abc} J_c, \quad [J_a, K_b] = i\varepsilon_{abc} K_c. \tag{174}$$

Note that these commutation relations differ from the algebra (97) of $SO(4)$ only in the commutator $[K_a, K_b]$.

It is instructive to compare the $J_a$'s and $K_a$'s of $SO(1,3)$ — see eqs. (170), (171) — to $J_a$'s and $K_a$'s of $SO(4)$ — see eqs. (93), (94). While all the $SO(4)$ generators are hermitian matrices, $K_a$'s of $SO(1,3)$ are not hermitian. Thus the representation of the $SO(1,3)$ group is not unitary. This happens due to non-compactness[15] of the $SO(1,3)$ group which materializes through unbounded values of the group parameter $\eta$ in (157). In contrast, possible values of all group parameters of $SO(4)$ fall within closed intervals. The $SO(4)$ group is compact and it has a finite unitary representations.

Let us define yet another set of $SO(1,3)$ generators

$$J_a^\pm \equiv \frac{1}{2}(J_a \pm iK_a), \quad a = 1,2,3, \tag{175}$$

which implies

$$J_a = J_a^+ + J_a^-, \quad K_a = \frac{1}{i}(J_a^+ - J_a^-). \tag{176}$$

The commutation relations for $J_a^\pm$ are

$$[J_a^\pm, J_b^\pm] = i\varepsilon_{abc} J_c^\pm, \quad [J_a^+, J_b^-] = 0. \tag{177}$$

This exactly coincides with the eq. (99). It implies that the $SO(1,3)$ group is locally isomorphic to the $SO(4)$ group as well as to the $SO(3) \otimes SO(3)$ group, and thus to the $SU(2) \otimes SU(2)$ group.

[15]Unitary representations of non-compact groups are infinite-dimensional.