Non-linear Sigma Model

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1 Math Introduction

- G group: T_a — hermitian generators
- H subgroup: S_a — hermitian generators

algebra $\mathcal{G}$ of the group G:

$$[T_a, T_b] = iF_{abc}T_c \tag{1}$$

normalization of the generators:

$$\mathrm{Tr}(T_aT_b) = \frac{1}{2}\delta_{ab} \tag{2}$$

algebra $\mathcal{H}$ of the subgroup H:

$$[S_a, S_b] = if_{abc}S_c \tag{3}$$

introducing non-$\mathcal{H}$ generators X_a:

$$\{T_a\}_{a=1}^{N} = \{X_1, \ldots, X_P, S_1, \ldots, S_M\}, \qquad P + M = N \tag{4}$$

consequences:

(2) $\Rightarrow$ *orthogonality* of S_a and X_b:

$$\mathrm{Tr}(S_aX_b) = 0 \tag{5}$$

(5) $\Rightarrow$

$$[S_a, X_b] = i\tilde{f}_{abc}X_c \tag{6}$$

(2), (3) $\Rightarrow$

$$if_{abc} = 2\mathrm{Tr}([S_a, S_b]S_c) \tag{7}$$

(2), (5) $\Rightarrow$

$$i\tilde{f}_{abc} = 2\mathrm{Tr}([S_a, X_b]X_c) \tag{8}$$

(7), (8) $\Rightarrow$

$$f_{aac} = f_{abb} = f_{aba} = 0, \qquad \tilde{f}_{abb} = 0 \tag{9}$$

with no summing over repeated indices.

(5) $\Rightarrow$

$$[X_a, X_b] = i\tilde{f}_{cba}S_c + i\hat{f}_{abd}X_d \tag{10}$$

some definitions:

- *coset space*: G/H
- *involutive automorphism* τ on G:

$$\forall g \in G : g \mapsto \tau(g) \in G \quad \text{such that} \quad \tau^2(g) = g \tag{11}$$

the symmetric coset space:

- The Eq. (10): if $\hat{f}_{abd} = 0, \forall a, b, d$, then G/H is called a *symmetric space*:

$$[X_a, X_b] = i\tilde{f}_{cba} S_c \tag{12}$$

- if G/H is the symmetric space then the mapping τ

$$\forall x \in \mathcal{G} : \tau(x) = \begin{cases} +x, & x \in \mathcal{H} \\ -x, & x \in \mathcal{G} - \mathcal{H} \end{cases} \tag{13}$$

  is the involutive automorphism on G. Indeed, $\tau^2(\pm x) = x \Rightarrow \tau^2(g) = g$, and

$$\begin{aligned} [\tau(S_a), \tau(S_b)] &= i f_{abc} \tau(S_c), & (14) \\ [\tau(S_a), \tau(X_b)] &= i \tilde{f}_{abc} \tau(X_c), & (15) \\ [\tau(X_a), \tau(X_b)] &= i \tilde{f}_{cba} \tau(S_c). & (16) \end{aligned}$$

summary:
in general

$$\begin{aligned} [\mathcal{H}, \mathcal{H}] &\subset \mathcal{H}, & (17) \\ [\mathcal{H}, \mathcal{G} - \mathcal{H}] &\subset \mathcal{G} - \mathcal{H}, & (18) \\ [\mathcal{G} - \mathcal{H}, \mathcal{G} - \mathcal{H}] &\subset \mathcal{G}. & (19) \end{aligned}$$

if G/H is the symmetric space

$$[\mathcal{G} - \mathcal{H}, \mathcal{G} - \mathcal{H}] \subset \mathcal{H} \tag{20}$$