Symmetries of the Standard Model

M. Gintner

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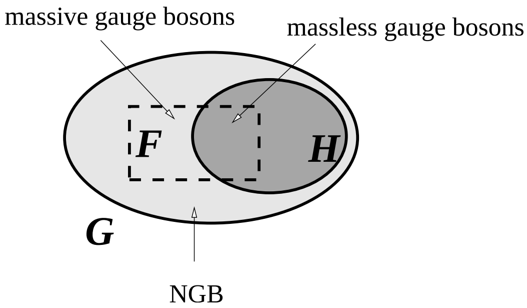
1 Why $SU(2)_L \times SU(2)_R$ global symmetry for the Higgs sector?

The gauge invariance as a fundamental principle for building the elweak Lagrangian enables to describe all elweak interactions under the same formalism. In addition it leads to predictions that have been verified experimentally. However, the local symmetry tolerates massless gauge fields only which is in contradiction with reality in all cases but the photon. Fortunately, there is a way to reconcile massive gauge bosons and gauge invariance: *the Higgs mechanism*[1] [F.Englert-R.Brout 64, P.W.Higgs 64, P.W.Higgs 66].

The Higgs mechanism is based on the *spontaneous symmetry breaking* (SSB): $G \to H$ where G is a global symmetry of the Lagrangian and H is a symmetry of its vacuum (ground state). The *Goldstone theorem* [J.Goldstone 61, J.Goldstone-A.Salam-S.Weinberg 62] implies that the SSB results in appearance of massless scalar fields, the so-called *Nambu-Goldstone bosons* (NGB). The number of NGB is equal to $\dim G - \dim H$, i.e. the number of "broken" generators of the group G. However, once $F \subset G$ is made local by introducing gauge fields the corresponding NGB's disappear from the spectrum while the $G - H$ gauge fields become massive and acquire longitudinal degrees of freedom. This is all depicted in the following figure:



Let us mention that massive fermions are also in conflict with gauge invariance under chiral groups. However, a gauge invariant mass term for fermions can be obtained by introducing Yukawa couplings in SSB.

Now we try to formulate the conditions that must be fulfiled in order to build the most general effective Lagrangian describing the Higgs mechanism that provides masses for the Z^0 and $W^\pm$ gauge bosons:

[1] The Higgs mechanism was first suggested to explain some collective density fluctuations in plasma which were produced by a finite range electromagnetic field (a massive photon) [P.W.Anderson 63]. Later it was generalized as a relativistic field theory and its renormalizability was proved [G.t'Hooft 71].

- A part of the SM must exhibit the SSB pattern $G \to H$, where G is a global symmetry and H its subgroup. Since we need masses for three gauge bosons we require that there would be at least 3 broken generators: $\dim K \geq 3$ where $K = G/H$ ($\dim K = \dim G - \dim H$).

- Three of broken generators must generate a local symmetry. The massive gauge bosons must be coupled to the corresponding three NGB's.

- The gauge group $SU(2)_L \times U(1)_Y$ has to be contained in G. The photon is a massless gauge boson thus $U(1)_{em} \subset H$. Therefore $\dim G \geq 4$, $\dim H \geq 1$.

- If the SSB bears the responsibility for the masses of gauge bosons then the masses should be given by the scale v of the SSB, $M_{W,Z} = \mathcal{O}(v)$. The value of v can be obtained from the Fermi constant G_F which is measured in the muon decay:

$$v^2 = \frac{1}{\sqrt{2} G_F}, \quad G_F \approx 1.17 \times 10^{-5} \text{ GeV}^{-2} \qquad \Rightarrow \qquad v \approx 246 \text{ GeV}$$

- The ρ parameter which is the relative strength of the charged weak currents to the neutral weak currents in the low energy effective Lagrangian has been measured to be very close to one,

$$\rho = \frac{\text{charged current}}{\text{neutral current}} \approx 1.$$

  The value of ρ depends on the structure of the Higgs sector. For example, when the group H contains the so-called *custodial symmetry* $SU(2)_V$ then $\rho = 1$. This is the case of the SM where at tree level

$$\rho = \frac{M_W^2}{M_Z^2 \cos^2 \theta_W} = 1.$$

  This value is affected by small radiative corrections

$$\rho = 1 + \underbrace{\mathcal{O}(g'^2)}_{\sim 0.01}$$

  due to $U(1)_Y$ coupling which explicitly breaks the custodial symmetry. The condition $SU(2)_V \subset H$ implies $\dim H \geq 3$.

If we consider $\dim K = 3$ (otherwise we would have to explain where the uneaten NGB's are) the constraints above lead to a unique choice of the groups G and H (see A.Dobado et al, Effective Lagrangians for the Standard Model, p.178 on):

$$G = SU(2)_L \times SU(2)_R, \quad H = SU(2)_V, \quad K = SU(2)_A,$$

where $SU(2)_V$ is a diagonal subgroup of $SU(2)_L \times SU(2)_R$ and $SU(2)_A$ is its axial[2] "subgroup". Therefore NGB's are pseudoscalar particles.

[2] $SU(2)_A$ is not a group even though it is often called that way in the physical literature.

2 Spontaneous Symmetry Breaking

Let G be a symmetry of a QM system

$$[H, U] = 0 \tag{1}$$

where H is the Hamiltonian of the system and $U(g)$ is a symmetry transformation, $g \in G$. Let $|1\rangle$ and $|2\rangle$ be states related by the symmetry transformation U

$$|2\rangle = U|1\rangle.$$

Then, if $|1\rangle$ is an eigenstate of H with an energy E, the $|2\rangle$ is also an eigenstate of H with the same energy

$$H|1\rangle = E|1\rangle \qquad \Rightarrow \qquad H|2\rangle = HU|1\rangle = UH|1\rangle = E|2\rangle.$$

All states that can be obtained by the symmetry transformations from $|1\rangle$ will be eigenstates of H with the eigenvalue E. In other words, G-invariant subspaces of the Hilbert space are degenerated in energy.

In QFT the Hamiltonian is a function of field operators $\varphi_i(x)$. Under the internal symmetry group G the field operators transform as follows

$$g \in G : \varphi_i(x) \to U(g)\varphi_i U(g)^\dagger.$$

For example, a particular transformation of the symmetry might relate the field operators of two different particle species

$$\varphi_2 = U(g)\varphi_1 U^\dagger(g).$$

Then also

$$a_2^\dagger(p) = U(g)a_1^\dagger(p)U^\dagger(g)$$

where $a_i^\dagger(p)$ are creation operators. When acting on the vacuum (the ground state of the theory) they generate one-particle states with the four-momenta p_i

$$|1\rangle = a_1^\dagger|0\rangle, \qquad |2\rangle = a_2^\dagger|0\rangle.$$

The one-particle states are eigenstates of the free-field Hamiltonian

$$H(\varphi_1, \varphi_2)|1\rangle = E(\vec{p}_1, m_1)|1\rangle, \qquad H(\varphi_1, \varphi_2)|2\rangle = E(\vec{p}_2, m_2)|2\rangle$$

where m_1, m_2 are masses of the two particles, $E(\vec{p}, m) = (\vec{p}^2 + m^2)^{1/2}$.

Now the question is whether the symmetry of the Hamiltonian guaranties that there is a symmetry transformation that relates the states $|1\rangle$ and $|2\rangle$ with the same momenta. If yes then the states would have the same energy and, consequently, the corresponding particles would have the same masses, $m_1 = m_2$. However,

$$U|1\rangle = Ua_1^\dagger|0\rangle = Ua_1^\dagger U^\dagger U|0\rangle = a_2^\dagger U|0\rangle.$$

Therefore, if the vacuum is symmetric under G the one-particle states are also connected with a symmetry transformation and the particle spectrum is degenerated

$$U|0\rangle = |0\rangle \qquad \Rightarrow \qquad U|1\rangle = |2\rangle \qquad \Rightarrow \qquad m_1 = m_2.$$

It is equivalent to the statement that if the Hamiltonian is symmetric and the masses are different then the vacuum is not invariant under G transformations

$$m_1 \neq m_2 \qquad \Rightarrow \qquad U|0\rangle \neq |0\rangle.$$

If the ground state of a theory has a lower symmetry than the symmetry of the Hamiltonian we say that the symmetry of the Hamiltonian is *spontaneously broken*. SSB has to be introduced once the particle spectrum does not reflect the symmetry of the theory.

3 $SU(2)_L \times SU(2)_R$

The $SU(2)$ group has two fundamental representations[3]

$$\text{covariant:} \quad \Phi' = U\Phi, \quad \Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \tag{2}$$

where U is 2×2 unitary matrix with $\det U = 1$, and

$$\text{contravariant (complex conj.):} \quad \bar{\chi}' = \bar{U}\bar{\chi}, \quad \bar{\chi} = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{\chi}_1 \\ \bar{\chi}_2 \end{pmatrix}, \tag{3}$$

where $\bar{U} \equiv U^* = (U^\dagger)^T$. The two representations are equivalent, i.e. there is a 2×2 regular matrix ϵ such that

$$\bar{U} = \epsilon^{-1} U \epsilon. \tag{4}$$

Let us denote $\bar{\epsilon} \equiv \epsilon^{-1}$. Then (4) implies that $\bar{\epsilon}\Phi$ transforms as $\bar{\Phi}$ and $\epsilon\bar{\Phi}$ transforms as Φ

$$(\bar{\epsilon}\Phi') = (\bar{\epsilon}U\epsilon)(\bar{\epsilon}\Phi) = \bar{U}(\bar{\epsilon}\Phi), \tag{5}$$

$$(\epsilon\bar{\Phi}') = (\epsilon\bar{U}\bar{\epsilon})(\epsilon\bar{\Phi}) = U(\epsilon\bar{\Phi}). \tag{6}$$

The U matrix can be expressed in an exponential form

$$U(\vec{\alpha}) = \exp(i\vec{\alpha}\vec{\tau}), \quad \vec{\tau} = \vec{\sigma}/2, \tag{7}$$

where $\vec{\alpha} = (\alpha^1, \alpha^2, \alpha^3)$ are real parameters, σ^a are the Pauli matrices. Then

$$\epsilon = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \bar{\epsilon} = -i\sigma^2 = -\epsilon = \epsilon^T. \tag{8}$$

Indeed, the contravariant representation in the exponential form reads

$$\bar{U}(\vec{\alpha}) = [\exp(i\vec{\alpha}\vec{\tau})]^* = \exp(-i\vec{\alpha}\vec{\tau}^*) = \exp(-i\vec{\alpha}\vec{\tau}^T). \tag{9}$$

For $\epsilon = i\sigma^2$ we can show that

$$\bar{\epsilon}(i\vec{\sigma})\epsilon = -i\vec{\sigma}^T. \tag{10}$$

Then the expression (9) can be rewritten as $\bar{U} = \bar{\epsilon}U\epsilon$ which coincides with (4).

Let us consider $G = SU(2)_L \times SU(2)_R$ group[4]. Certainly, any $SU(2)$ representation is a representation of G once we decide which $SU(2)$ subgroup of G it represents. For example, under G an $SU(2)$ doublet Φ can transform as follows

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}: \quad \Phi \xrightarrow{G} \Phi' = L\Phi, \quad L \in SU(2)_L \tag{11}$$

[3]The factor of $1/\sqrt{2}$ in Φ is introduced for later convenience. This normalization is common in the literature when the Higgs sector with complex scalar field doublet(s) is defined.

[4]The subscripts on the $SU(2)$ symbols have been introduced to distinguish between the two $SU(2)$ components of the direct product. At this point there is no particular reason for which the subscripts have been named L and R. Any two different symbols would do as well.

and we would call it the $SU(2)_L$ doublet. Then $\tilde{\Phi} \equiv \epsilon\Phi^*$ also transforms under $SU(2)_L$

$$\tilde{\Phi} \equiv \epsilon\Phi^* = \frac{1}{\sqrt{2}}\left(\begin{array}{c} \varphi_2^* \\ -\varphi_1^* \end{array}\right): \quad \tilde{\Phi} \stackrel{G}{\to} \tilde{\Phi}' = L\tilde{\Phi} \tag{12}$$

In the same way we can have $SU(2)_R$ doublets. Let Υ be an $SU(2)_R$ doublet transforming under contravariant representation

$$\Upsilon = \frac{1}{\sqrt{2}}\left(\begin{array}{c} \xi_1 \\ \xi_2 \end{array}\right): \quad \Upsilon \stackrel{G}{\to} \Upsilon' = \bar{R}\Upsilon, \quad R \in SU(2)_R \tag{13}$$

Then also $\tilde{\Upsilon} \equiv \bar{\epsilon}\Upsilon^*$ is a contravariant $SU(2)_R$ doublet

$$\tilde{\Upsilon} \equiv \bar{\epsilon}\Upsilon^* = \frac{1}{\sqrt{2}}\left(\begin{array}{c} -\xi_2^* \\ \xi_1^* \end{array}\right): \quad \tilde{\Upsilon} \stackrel{G}{\to} \tilde{\Upsilon}' = \bar{R}\tilde{\Upsilon} \tag{14}$$

If in (13) we identify $\xi_1 = \varphi_2^*$ and $\xi_2 = \varphi_1$ we can construct a 2×2 complex matrix

$$M \equiv \sqrt{2}\left(\tilde{\Phi}, \Phi\right) = \sqrt{2}\left(\begin{array}{c} \Upsilon^T \\ \tilde{\Upsilon}^T \end{array}\right) = \left(\begin{array}{cc} \varphi_2^* & \varphi_1 \\ -\varphi_1^* & \varphi_2 \end{array}\right) \tag{15}$$

which under $SU(2)_L \times SU(2)_R$ transforms in the following way

$$M \stackrel{G}{\to} M' = LMR^\dagger, \tag{16}$$

where L and R are covariant representations of $SU(2)_L$ and $SU(2)_R$, respectively. Let us express φ_1 and φ_2 in terms of their real components $\pi^1, \pi^2, \pi^3, \sigma$: $\varphi_1 = \pi^2 + i\pi^1$, $\varphi_2 = \sigma - i\pi^3$. Then M can be written as a linear combination of the unit matrix $\mathbf{I}^{(2)}$ and the Pauli matrices σ^i

$$M = \mathbf{I}^{(2)}\sigma + i\vec{\pi}\vec{\sigma} = \left(\begin{array}{cc} \sigma + i\pi^3 & \pi^2 + i\pi^1 \\ -\pi^2 + i\pi^1 & \sigma - i\pi^3 \end{array}\right). \tag{17}$$

Some relations that will come handy later on:

$$M^\dagger = \sqrt{2}\left(\begin{array}{c} \tilde{\Phi}^\dagger \\ \Phi^\dagger \end{array}\right) = \left(\begin{array}{cc} \varphi_2 & -\varphi_1 \\ \varphi_1^* & \varphi_2^* \end{array}\right) = \left(\begin{array}{cc} \sigma - i\pi^3 & -\pi^2 - i\pi^1 \\ \pi^2 - i\pi^1 & \sigma + i\pi^3 \end{array}\right) = \mathbf{I}^{(2)}\sigma - i\vec{\pi}\vec{\sigma}, \tag{18}$$

$$\Phi^\dagger\Phi = \tilde{\Phi}^\dagger\tilde{\Phi} = \frac{1}{2}(|\varphi_1|^2 + |\varphi_2|^2) = \frac{1}{2}[(\sigma)^2 + (\vec{\pi})^2], \quad \tilde{\Phi}^\dagger\Phi = \Phi^\dagger\tilde{\Phi} = 0, \tag{19}$$

$$M^\dagger M = 2\left(\begin{array}{cc} \tilde{\Phi}^\dagger\tilde{\Phi} & \tilde{\Phi}^\dagger\Phi \\ \Phi^\dagger\tilde{\Phi} & \Phi^\dagger\Phi \end{array}\right) = \mathbf{I}^{(2)}[(\sigma)^2 + (\vec{\pi})^2], \tag{20}$$

$$\mathrm{Tr}(M^\dagger M) = 2(\tilde{\Phi}^\dagger\tilde{\Phi} + \Phi^\dagger\Phi) = 4\,\Phi^\dagger\Phi = 2(|\varphi_1|^2 + |\varphi_2|^2) = 2[(\sigma)^2 + (\vec{\pi})^2]. \tag{21}$$

The representation of the $SU(2)_L \times SU(2)_R$ group is homomorphic to the fundamental representation of the $SO(4)$ group. Let us clarify it. It was shown above that

$$\frac{1}{2}\mathrm{Tr}(M^\dagger M) = (\sigma)^2 + (\vec{\pi})^2.$$

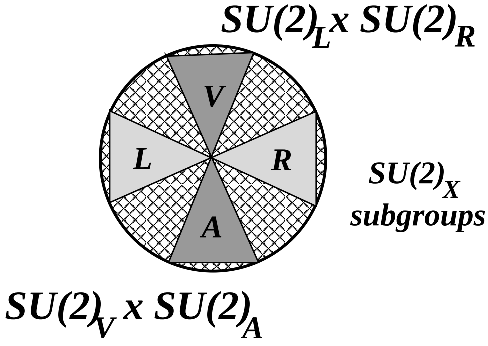
Applying (16) we get that $\mathrm{Tr}(M^\dagger M)$ is invariant under $SU(2)_L \times SU(2)_R$ transformations. Thus if $\omega^T = (\pi^1, \pi^2, \pi^3, \sigma)^T$ is an $SO(4)$ vector then the transformations (16) leave invariant the same quantity as $SO(4)$ rotations; namely the norm of the vector, $|\omega| = [(\sigma)^2 + (\vec{\pi})^2]^{1/2}$. The mapping is homomorphism, though, since $(L, R) \in SU(2)_L \times SU(2)_R$ leads to the same change of $\vec{\omega}$ as $(-L, -R)$. Thus, at least two $SU(2)_L \times SU(2)_R$ transformations correspond to a single $SO(4)$ rotation.

The $G = SU(2)_L \times SU(2)_R$ group contains several subgroups. They are listed in the table below. There, a general element $g \in G$ is parameterized as $g(\vec{\alpha}_L, \vec{\alpha}_R) = \exp[i(\alpha^a_L T^a_L + \alpha^b_R T^b_R)]$, $a, b = 1, 2, 3$, where $T^a_{L,R}$ are generators of the $SU(2)_{L,R}$ groups. The generators hold the algebra $[T^a_{L,R}, T^b_{L,R}] = i\varepsilon^{abc} T^c_{L,R}$, $[T^a_L, T^b_R] = 0$. Thus $g(\vec{\alpha}_L, \vec{\alpha}_R) = g_L(\vec{\alpha}_L) g_R(\vec{\alpha}_R)$ where $g_{L,R} \in SU(2)_{L,R}$.

subgroup H	$g(\vec{\alpha}_L, \vec{\alpha}_R)$	note
$SU(2)_L$	$\vec{\alpha}_R = 0$	
$SU(2)_R$	$\vec{\alpha}_L = 0$	
$SU(2)_V$	$\vec{\alpha}_L = \vec{\alpha}_R$	$[T^a_L + T^a_R, T^b_L + T^b_R] = i\varepsilon^{abc}(T^c_L + T^c_R)$
$U(1)_{L3}$	$\vec{\alpha}_L = (0, 0, \alpha_L), \vec{\alpha}_R = 0$	$i)$ also $gHg^{-1}, g \in G$ $ii)$ $\subset SU(2)_L$
$U(1)_{R3}$	$\vec{\alpha}_L = 0, \vec{\alpha}_R = (0, 0, \alpha_R)$	$i)$ also $gHg^{-1}, g \in G$ $ii)$ $\subset SU(2)_R$
$U(1)_{L3} \times U(1)_{R3}$	$\vec{\alpha}_L = (0, 0, \alpha_L), \vec{\alpha}_R = (0, 0, \alpha_R)$	also $g_1(U(1)_{L3})g_1^{-1} \times g_2(U(1)_{R3})g_2^{-1}$, $g_1, g_2 \in G$
$U(1)_{V3}$	$\vec{\alpha}_L = (0, 0, \alpha), \vec{\alpha}_R = (0, 0, \alpha)$	$i)$ also $gHg^{-1}, g \in SU(2)_V$ $ii)$ $\subset SU(2)_V$ $iii)$ $\subset U(1)_{L3} \times U(1)_{R3}$
$U(1)_{A3}$	$\vec{\alpha}_L = (0, 0, -\beta), \vec{\alpha}_R = (0, 0, \beta)$	$i)$ also $gHg^{-1}, g \in SU(2)_V$ $ii)$ $\subset U(1)_{L3} \times U(1)_{R3}$
$U(1)_{V3} \times U(1)_{A3}$	$\vec{\alpha}_L = (0, 0, \alpha - \beta), \vec{\alpha}_R = (0, 0, \alpha + \beta)$	$= U(1)_{L3} \times U(1)_{R3}$

(22)

The set of elements $g(\vec{\alpha}_L, \vec{\alpha}_R) \in G$ where $\vec{\alpha}_L = -\vec{\alpha}_R$ is usually denoted as $SU(2)_A$. It does not comprise a group[5] even though $SU(2)_L \times SU(2)_R = SU(2)_A \times SU(2)_V$. The direct product on the r.h.s. means that any $g \in G$ can be decomposed in the form $g = \xi h$ where $\xi \in SU(2)_A$, $h \in SU(2)_V$. The structure of the $SU(2)_L \times SU(2)_R$ group in terms of its $SU(2)_X$ subgroups, where $X = L, R, A, V$, is depicted in the following figure:



[5]The generators of $SU(2)_A$ do not form an algebra: $[T^a_R - T^a_L, T^b_R - T^b_L] = i\varepsilon^{abc}(T^c_R + T^c_L)$

Only the unit element of the group is common to all the four $SU(2)$ subsets.

4 Theory based on the $SU(2)_L \times SU(2)_R$ global symmetry

In the following we are going to build Lagrangians using hermitian Lorentz invariant field operators up to mass dimension of 4. It implies that coupling constants will be of dimension ≥ 0. Recall that the dimensions of the individual Lorentz types of fields are

spin	0	1/2	1
dim	1	3/2	1

In addition, the mass dimension of ∂_μ is 1 and the dimension of the Lagrangian density is 4.

4.1 Higgs Lagrangian

Let us consider the following Lagrangian

$$L_H = \frac{1}{4}\mathrm{Tr}(\partial_\mu M \partial^\mu M^\dagger) + \underbrace{\frac{\mu^2}{4}\mathrm{Tr}(M^\dagger M) - \frac{\lambda}{16}[\mathrm{Tr}(M^\dagger M)]^2}_{-V(M)}, \tag{23}$$

where $M(x)$ is a 2×2 matrix of scalar fields defined as in (15) and (17)

$$M(x) = \mathbf{I}^{(2)}\sigma(x) + i\vec{\pi}(x)\vec{\sigma}, \qquad M(x) \stackrel{SU(2)_L\times SU(2)_R}{\longrightarrow} LM(x)R^\dagger.$$

The Lagrangian L_H is invariant under global $SU(2)_L \times SU(2)_R$ transformations (16). It can be expressed in terms of other objects introduced above

$$\begin{aligned} L_H &= (\partial_\mu\Phi^\dagger)(\partial^\mu\Phi) + \mu^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2 \\ &= \frac{1}{2}(\partial_\mu\vec{\pi})(\partial^\mu\vec{\pi}) + \frac{1}{2}(\partial_\mu\sigma)(\partial^\mu\sigma) + \frac{\mu^2}{2}[(\vec{\pi})^2 + (\sigma)^2] - \frac{\lambda}{4}[(\vec{\pi})^2 + (\sigma)^2]^2. \end{aligned} \tag{24}$$

The potential $V(M)$ has minima at M satisfying the condition

$$\frac{1}{2}\mathrm{Tr}(M^\dagger M) = \frac{\mu^2}{\lambda} \equiv v^2. \tag{25}$$

Obviously, the set of all the minima is a hypersurface closed under $SU(2)_L \times SU(2)_R$ transformations. Let us choose one of the minima to become the physical vacuum

$$M_{vac} = v\mathbf{I}^{(2)} = \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}. \tag{26}$$

The vacuum possesses less symmetry than the Lagrangian. It is invariant under $SU(2)_V$ only

$$M'_{vac} = UM_{vac}U^\dagger = vUU^\dagger = v\mathbf{I}^{(2)} = M_{vac}. \tag{27}$$

Recall that this discrepancy between the symmetry of the Lagrangian and the symmetry of its vacuum is called the spontaneous symmetry breaking. At the same time the $SU(2)_V$ transformations interconnect the fields of the $\vec{\pi}$ triplet

$$SU(2)_V: \quad \vec{\pi}\vec{\sigma} \to \vec{\pi}'\vec{\sigma} = U(\vec{\pi}\vec{\sigma})U^\dagger. \tag{28}$$

Since $SU(2)_V$ is both the symmetry of the Lagrangian as well as the symmetry of the vacuum it is reasonable to expect that the masses of all the π fields are the same. That is what we find when we reparameterize the fields of the matrix M as deviations from the vacuum (26)

$$M = \sqrt{2}[v + h(x)]\mathbf{I}^{(2)} + i\sqrt{2}\vec{\pi}(x)\vec{\sigma} \tag{29}$$

Upon substituting (29) into the Lagrangian (23) we find out that the triplet $\vec{\pi}(x)$ represents massless particles called *Nambu-Goldstone bosons* while $h(x)$, called the *Higgs boson*, has a mass $m_h = \sqrt{2}\mu$. Beside that we can find the self-interaction terms among the NGB's as well as the interaction terms between the NGB's and the Higgs boson. All these interactions are governed by the coupling constant λ.

We have shown that the $SU(2)_L \times SU(2)_R$ group is locally isomorphic to the $SO(4)$ group. Out of the four scalar fields $\vec{\pi}(x)$, $\sigma(x)$ we can form an $SO(4)$ vector $\omega^T = (\pi^1, \pi^2, \pi^3, \sigma)^T$ which transforms as

$$\omega' = O(g)\omega, \qquad g \in SO(4)$$

where $O(g)$ is a 4×4 real matrix which fulfils the following conditions

$$O^T O = \mathbf{I}^{(4)}, \qquad \det O = 1.$$

The Lagrangian (23) expressed in terms of ω is

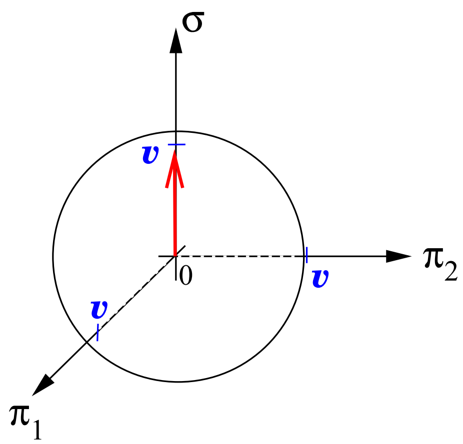
$$L_H = \frac{1}{2}(\partial_\mu \omega^T)(\partial^\mu \omega) + \frac{\mu^2}{2}(\omega^T\omega) - \frac{\lambda}{4}(\omega^T\omega)^2. \tag{30}$$

The Lagrangian L_H is obviously invariant under $SO(4)$ transformations. It has a minimum energy at $\omega^T\omega = v^2$.

Four-dimensional rotations are difficult to visualize. However, it is very easy to write down $SO(3)$-invariant Lagrangian in complete analogy to (30) considering only triplet of scalar fields $\omega = (\pi^1, \pi^2, \sigma)$. The Lagrangian is

$$L_H = (\partial_\mu \omega^T)(\partial^\mu \omega) + \mu^2(\omega^T\omega) - \lambda(\omega^T\omega)^2.$$

The minimum energy of the system lies on the surface of the 3-dim sphere in the (π^1, π^2, σ)-space and we can choose the vacuum as $\omega^T_{vac} = (0, 0, v)^T$:



The chosen vacuum is invariant under $SO(2)$ rotations around the σ-axis while (π^1, π^2) form the $SO(2)$-invariant subspace. Thus it is no surprise that the $\pi^1(x)$ and $\pi^2(x)$ fields have the same mass, equal to zero, and that the $h(x)$ field, which is a σ-aligned deviation from the vacuum ω_{vac}, possesses a non-zero mass.

4.2 Vector Field Lagrangian

Should some vector fields appear in the theory which is invariant under the global $SU(2)_L \times SU(2)_R$ transformations one possibility would be two vector field triplets: an $SU(2)_L$ triplet $\{\mathcal{L}_\mu^a\}_{a=1}^3$, and an $SU(2)_R$ triplet $\{\mathcal{R}_\mu^a\}_{a=1}^3$, which would transform by the adjoint representations of $SU(2)_L$ and $SU(2)_R$, respectively,

$$\begin{aligned} \mathcal{L}'_\mu &= U_L \mathcal{L}_\mu U_L^\dagger, \qquad \mathcal{L}_\mu \equiv \mathcal{L}_\mu^a \tau^a, & (31) \\ \mathcal{R}'_\mu &= U_R \mathcal{R}_\mu U_R^\dagger, \qquad \mathcal{R}_\mu \equiv \mathcal{R}_\mu^a \tau^a. & (32) \end{aligned}$$

For each triplet we can define the field-strength tensor

$$\begin{aligned} \mathcal{L}_{\mu\nu} &= \partial_\mu \mathcal{L}_\nu - \partial_\nu \mathcal{L}_\mu + i g_L [\mathcal{L}_\mu, \mathcal{L}_\nu], & (33) \\ \mathcal{R}_{\mu\nu} &= \partial_\mu \mathcal{R}_\nu - \partial_\nu \mathcal{R}_\mu + i g_R [\mathcal{R}_\mu, \mathcal{R}_\nu], & (34) \end{aligned}$$

where $g_{L,R}$ are coupling constants related to possible self-interactions of the vector fields. Under $SU(2)_L \times SU(2)_R$ the field-strength tensors transform as follows

$$\begin{aligned} \mathcal{L}_{\mu\nu} &\rightarrow U_L \mathcal{L}_{\mu\nu} U_L^\dagger, & (35) \\ \mathcal{R}_{\mu\nu} &\rightarrow U_R \mathcal{R}_{\mu\nu} U_R^\dagger, & (36) \end{aligned}$$

Now, the $SU(2)_L \times SU(2)_R$ globally invariant vector field Lagrangian reads

$$L_V = -\frac{1}{4}\mathrm{Tr}(\mathcal{L}_{\mu\nu}\mathcal{L}^{\mu\nu}) - m_\mathcal{L}^2 \mathrm{Tr}(\mathcal{L}_\mu \mathcal{L}^\mu) - \frac{1}{4}\mathrm{Tr}(\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}) - m_\mathcal{R}^2 \mathrm{Tr}(\mathcal{R}_\mu \mathcal{R}^\mu). \qquad (37)$$

The ground state of the Lagrangian corresponds to "zero-field" situation, $\mathcal{L}_\mu = \mathcal{R}_\mu = 0$. Thus the vector-field vacuum possesses the same $SU(2)_L \times SU(2)_R$ global symmetry as the Lagrangian L_V.

Let us break down individual terms of the Lagrangian. First of all, there is a mass term

$$m_\mathcal{X}^2 \mathrm{Tr}(\mathcal{X}_\mu \mathcal{X}^\mu) = \frac{1}{2} m_\mathcal{X}^2 \vec{\mathcal{X}}_\mu \vec{\mathcal{X}}^\mu = \frac{1}{2} m_\mathcal{X}^2 [\mathcal{X}_\mu^3 \mathcal{X}^{3\mu} + (\mathcal{X}_\mu^+)^\dagger \mathcal{X}^{+\mu} + (\mathcal{X}_\mu^-)^\dagger \mathcal{X}^{-\mu}],$$

where $\mathcal{X}_\mu^\pm \equiv (\mathcal{X}_\mu^1 \mp i\mathcal{X}_\mu^2)$ and $\mathcal{X}_\mu^3$ are charged and neutral vector bosons of the given triplet, respectively, and $m_\mathcal{X}$ is the mass of its constituents. We can see that the masses of each triplet are degenerate as we would expect from symmetry considerations. Further,

$$\begin{aligned} \mathrm{Tr}(\mathcal{X}_{\mu\nu}\mathcal{X}^{\mu\nu}) = \; & \mathrm{Tr}[(\partial_\mu \mathcal{X}_\nu - \partial_\nu \mathcal{X}_\mu)(\partial^\mu \mathcal{X}^\nu - \partial^\nu \mathcal{X}^\mu)] \\ & + i g_\mathcal{X} \mathrm{Tr}(\{\partial_\mu \mathcal{X}_\nu - \partial_\nu \mathcal{X}_\mu, [\mathcal{X}^\mu, \mathcal{X}^\nu]\}) \\ & - g_\mathcal{X}^2 \mathrm{Tr}([\mathcal{X}_\mu, \mathcal{X}_\nu][\mathcal{X}^\mu, \mathcal{X}^\nu]). & (38) \end{aligned}$$

Here, the first line term represents the kinetic energy of the triplet $\mathcal{X}_\mu$. The second line[6] is the triple self-interactions of the vector fields and the third line is their quadruple self-interactions.

4.3 Fermion Lagrangian

Now we try to include fermions to the global $SU(2)_L \times SU(2)_R$ invariant theory. For example, let us introduce fermions that form $SU(2)_L$ and $SU(2)_R$ doublets

$$SU(2)_L \times SU(2)_R: \; \Psi'_I = U_I \Psi_I, \quad \Psi_I \equiv \begin{pmatrix} u_I \\ d_I \end{pmatrix}, \quad I = L, R, \qquad (39)$$

[6] Note that the symbol $\{\,,\}$ in the second line stands for the anticommutator.

where $U_I \in SU(2)_I$, and u_I, d_I are Dirac spinors of fermions forming the $SU(2)_I$ doublet. It means that Ψ_L is invariant under $SU(2)_R$, and *vice versa*. Then the following free fermion Lagrangian can be built[7]

$$L_f = \bar{\Psi}_L(i\,\partial\!\!\!/ - m_L)\Psi_L + \bar{\Psi}_R(i\,\partial\!\!\!/ - m_R)\Psi_R. \tag{40}$$

This means that there are two sorts of fermions: L-fermions u_L and d_L and R-fermions u_R and d_R. The Lagrangian L_f has the ground state at $u_L = d_L = u_R = d_R = 0$ and thus the $SU(2)_L \times SU(2)_R$ global symmetry is also a symmetry of the fermion vacuum. Therefore the symmetry is reflected also in the particle spectrum: all fermions of the doublet Ψ_I have the same masses, m_I.

4.4 Interactions of Vector Fields to Fermions

The globally invariant $SU(2)_L \times SU(2)_R$ interactions between the fermions and the vector bosons introduced above are as follows

$$L_{fV} = -g_L^{fV}\bar{\Psi}_L\gamma^\mu\mathcal{L}_\mu\Psi_L - g_R^{fV}\bar{\Psi}_R\gamma^\mu\mathcal{R}_\mu\Psi_R, \tag{41}$$

where g_L^{fV} and g_R^{fV} are coupling constants. The signs are a matter of convention. We can see that the symmetry ties up interactions of several fields. The detailed structure of (41) reveals when we substitute there

$$\begin{aligned}\bar{\Psi}\gamma^\mu\mathcal{X}_\mu\Psi \;\; = \;\; & \frac{1}{2}\bar{u}\gamma^\mu\mathcal{X}_\mu^3 u - \frac{1}{2}\bar{d}\gamma^\mu\mathcal{X}_\mu^3 d \\ & + \frac{1}{\sqrt{2}}(\bar{u}\gamma^\mu\mathcal{X}_\mu^+ d + \bar{d}\gamma^\mu\mathcal{X}_\mu^- u).\end{aligned} \tag{42}$$

In deriving it we used $\vec{\mathcal{X}}_\mu\vec{\tau} = \mathcal{X}_\mu^3\tau^3 + (\mathcal{X}_\mu^+\tau^+ + \mathcal{X}_\mu^-\tau^-)/\sqrt{2}$ where $\tau^\pm = \tau^1 \pm i\tau^2$.

4.5 Interactions to the Higgs Sector

Let us begin with the interactions between vector bosons and complex scalar doublet matrix M

$$\begin{aligned}L_{VH} \;\; = \;\; & \frac{1}{4}g_L^{VH}\mathrm{Tr}[(\partial^\mu M^\dagger)\mathcal{L}_\mu M \pm M^\dagger\mathcal{L}_\mu(\partial^\mu M)] \\ & + \frac{1}{4}g_R^{VH}\mathrm{Tr}[(\partial^\mu M)\mathcal{R}_\mu M^\dagger \pm M\mathcal{R}_\mu(\partial^\mu M^\dagger)] \\ & + \frac{1}{4}g_{LL}^{VH}\mathrm{Tr}(M^\dagger\mathcal{L}_\mu\mathcal{L}^\mu M) \\ & + \frac{1}{4}g_{RR}^{VH}\mathrm{Tr}(M\mathcal{R}_\mu\mathcal{R}^\mu M^\dagger) \\ & + \frac{1}{2}g_{RL}^{VH}\mathrm{Tr}(M\mathcal{R}_\mu M^\dagger\mathcal{L}^\mu).\end{aligned} \tag{43}$$

There are five new coupling constants associated with this type of interactions. The factors of 1/4 and 1/2 have been introduced for the sake of later convenience. In the first two lines the plus signs apply when $g_{L,R}^{VH}$ are real numbers. The minus signs apply when these constants are pure imaginary numbers. The other coupling constants in the last three lines must be real numbers.

[7] The dimension of the fermion field (3/2) is too high to allow for fermion field self-interactions which would fulfil the criteria formulated in the beginning of this Section.

These interactions connect the vector fields to the spontaneously broken Higgs sector. Thus they mediate the effect of the SSB on the spectrum of the vector particles. Let us substitute $M = [v + h(x)]\mathbf{I}^{(2)} + i\vec{\pi}(x)\vec{\sigma}$ into (43). From the last three lines of L_{VH} and (37) we obtain vector field mass-like terms

$$L_{Vmass} = \frac{1}{2}\frac{\tilde{g}^{VH}_{LL}v^2}{4}\vec{\mathcal{L}}_\mu\vec{\mathcal{L}}^\mu + \frac{1}{2}\frac{\tilde{g}^{VH}_{RR}v^2}{4}\vec{\mathcal{R}}_\mu\vec{\mathcal{R}}^\mu + \frac{g^{VH}_{RL}v^2}{4}\vec{\mathcal{R}}_\mu\vec{\mathcal{L}}^\mu \tag{44}$$

where

$$\frac{\tilde{g}^{VH}_{LL}v^2}{4} = \frac{g^{VH}_{LL}v^2}{4} + m^2_{\mathcal{L}}, \qquad \frac{\tilde{g}^{VH}_{RR}v^2}{4} = \frac{g^{VH}_{RR}v^2}{4} + m^2_{\mathcal{R}} \tag{45}$$

If we use $\vec{\mathcal{X}}_\mu\vec{\mathcal{Y}}^\mu = \mathcal{X}^3_\mu\mathcal{Y}^{3\mu} + \mathcal{X}^+_\mu\mathcal{Y}^{-\mu} + \mathcal{X}^-_\mu\mathcal{Y}^{+\mu}$ then

$$\begin{aligned} L_{Vmass} \quad = \quad & \frac{1}{2}(\mathcal{L}^3_\mu, \mathcal{R}^3_\mu)\overbrace{\frac{v^2}{4}\begin{pmatrix} \tilde{g}^{VH}_{LL} & g^{VH}_{RL} \\ g^{VH}_{RL} & \tilde{g}^{VH}_{RR} \end{pmatrix}}^{\mathcal{M}^2}\begin{pmatrix} \mathcal{L}^{3\mu} \\ \mathcal{R}^{3\mu} \end{pmatrix} \\ & +\frac{1}{2}(\mathcal{L}^-_\mu, \mathcal{R}^-_\mu)\frac{v^2}{4}\begin{pmatrix} \tilde{g}^{VH}_{LL} & g^{VH}_{RL} \\ g^{VH}_{RL} & \tilde{g}^{VH}_{RR} \end{pmatrix}\begin{pmatrix} \mathcal{L}^{+\mu} \\ \mathcal{R}^{+\mu} \end{pmatrix} \\ & +\frac{1}{2}(\mathcal{L}^+_\mu, \mathcal{R}^+_\mu)\frac{v^2}{4}\begin{pmatrix} \tilde{g}^{VH}_{LL} & g^{VH}_{RL} \\ g^{VH}_{RL} & \tilde{g}^{VH}_{RR} \end{pmatrix}\begin{pmatrix} \mathcal{L}^{-\mu} \\ \mathcal{R}^{-\mu} \end{pmatrix} \end{aligned}$$

Beside that we can get the interaction terms for the couplings of vector fields to the Higgs boson and to the NGB's from (43).

The matrix $\mathcal{M}^2$ is symmetric and thus diagonalizable (see Appendix). To diagonalize it the coordinates $\{\mathcal{L}, \mathcal{R}\}$ of the doublet in an orthogonal basis has to be transformed to the basis formed by the eigenvectors of the matrix. In its diagonal form the matrix reads $\mathcal{M}^2 = \mathrm{diag}(\mu_1^2, \mu_2^2)$ where μ_1^2, μ_1^2 are eigenvalues of $\mathcal{M}^2$

$$\mu^2_{1,2} = \frac{v^2}{8}\left[(\tilde{g}^{VH}_{LL} + \tilde{g}^{VH}_{RR}) \pm \sqrt{(\tilde{g}^{VH}_{LL} - \tilde{g}^{VH}_{RR})^2 + 4(g^{VH}_{RL})^2}\right] \tag{46}$$

The eigenvectors of $\mathcal{M}^2$ form an orthogonal basis. Let us denote the coordinates of the doublet in the basis of the eigenvectors as $\{\mathcal{T}, \mathcal{U}\}$. Then the transition from $\{\mathcal{L}, \mathcal{R}\}$ to $\{\mathcal{T}, \mathcal{U}\}$ is governed by an orthogonal transformation matrix which can be parameterized by a single angle $\theta = \theta(\tilde{g}^{VV}_{LL}, \tilde{g}^{VV}_{RR}, g^{VV}_{RL})$

$$\begin{pmatrix} \mathcal{L} \\ \mathcal{R} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \mathcal{T} \\ \mathcal{U} \end{pmatrix}, \qquad \begin{pmatrix} \mathcal{T} \\ \mathcal{U} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \mathcal{L} \\ \mathcal{R} \end{pmatrix}. \tag{47}$$

When (47) is plugged in (44) we end up with

$$L_{Vmass} = \frac{1}{2}\mu_1^2\vec{\mathcal{T}}_\mu\vec{\mathcal{T}}^\mu + \frac{1}{2}\mu_2^2\vec{\mathcal{U}}_\mu\vec{\mathcal{U}}^\mu. \tag{48}$$

This means that detecting energy and momentum of vector particles we would see three vector bosons $\mathcal{T}^\pm, \mathcal{T}^3$ of the mass μ_1 and three vector bosons $\mathcal{U}^\pm, \mathcal{U}^3$ of the mass μ_2 rather than $\mathcal{L}$'s and $\mathcal{R}$'s. Recall from (47) that the states $\mathcal{T}^\pm, \mathcal{T}^3$ and $\mathcal{U}^\pm, \mathcal{U}^3$ are linear combinations of $\mathcal{L}^\pm, \mathcal{L}^3$ and $\mathcal{R}^\pm, \mathcal{R}^3$. The former are called the *mass eigenstates* while the latter are called the *interaction* *eigenstates.* If we wish to study physics of the mass eigenstates we have to substitute the first relation of (47) in the vector boson Lagrangians (38) and (42).

The mixing angle θ for a given combination of coupling constants can be found from (see Appendix)

$$\sin\theta = \frac{|g_{RL}|}{\sqrt{g_{RL}^2 + (\tilde{g}_{LL} - 4\mu_2^2/v^2)^2}}, \qquad \cos\theta = \frac{(2\mu_2/v)^2 - \tilde{g}_{LL}}{g_{RL}} \sin\theta. \tag{49}$$

The fact that transformation (47) should take us from (44) to (48) results in the relation

$$(\tilde{g}_{LL} - \tilde{g}_{RR}) \sin(2\theta) = 2 g_{RL} \cos(2\theta)$$

or

$$\tan(2\theta) = \frac{2 g_{RL}}{\tilde{g}_{LL} - \tilde{g}_{RR}}. \tag{50}$$

For the sake of brevity we have dropped the superscript VH from all coupling constants in (49) and (50).

The globally $SU(2)_L \times SU(2)_R$ invariant interactions between the fermion $SU(2)_{L,R}$ doublets and the Higgs sector are described by the interaction Lagrangian

$$L_{fH} = c_f (\bar{\Psi}_L M \Psi_R + \bar{\Psi}_R M^\dagger \Psi_L) \tag{51}$$

which in terms of Φ and $\tilde{\Phi}$ reads

$$L_{fH} = \sqrt{2} c_f [\bar{\Psi}_L (\tilde{\Phi} u_R + \Phi d_R) + (\bar{u}_R \tilde{\Phi}^\dagger + \bar{d}_R \Phi^\dagger) \Psi_L]$$

where Φ and $\tilde{\Phi}$ are scalar $SU(2)$ doublets defined in (11) and (12) and c_f is a coupling constant.

Again, these interactions mediate the effect of the SSB on fermions. If we substitute $M = [v + h(x)]\mathbf{I}^{(2)} + i\vec{\pi}(x)\vec{\sigma}$ in (51) we find, along with the interactions of fermions to the Higgs boson and to the NGB's, terms bilinear in fermion fields

$$L_{ff} = v c_f (\bar{u}_L u_R + \bar{u}_R u_L) + v c_f (\bar{d}_R d_L + \bar{d}_L d_R) \tag{52}$$

which are difficult to interpret since in general they connect different particle species, u_L to u_R, and d_L to d_R.

5 Chiral Fermions

5.1 Parity: Weyl *vs* Dirac Representations

Fundamental fermion fields possess a spin 1/2 which doubles their degrees of freedom comparing to spinless particles. Thus the fermions have to be described by two-component fields, at least.

The fermion field is a representation of the Lorentz group $SO(1,3)$. The $SO(1,3)$ group[8] is locally isomorphic to the $SU(2) \times SU(2)$ group. Therefore, its representations can be built as a direct sum of two $SU(2)$ representations and they can be labelled (j_+, j_-), where $j_\pm$ are "spins" of the individual $SU(2)$ components of the direct product. The dimension of the representation (j_+, j_-) is $(2j_+ + 1)(2j_- + 1)$.

[8]The $SO(1,3)$ group is a group of 4-dim real matrices of a unitary determinant transforming a vector (x_0, x_1, x_2, x_3) in the way which maintains $(x_0)^2 - (x_1)^2 - (x_2)^2 - (x_3)^2$ unchanged.

There are two two-dimensional representations of the Lorentz group: $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$. They transform complex doublets ξ^+ and ξ^-, respectively, called the *Weyl spinors*

$$(\frac{1}{2},0): \ \xi^+ \stackrel{SO(1,3)}{\longrightarrow} \Lambda^+\xi^+, \qquad (0,\frac{1}{2}): \ \xi^- \stackrel{SO(1,3)}{\longrightarrow} \Lambda^-\xi^-, \tag{53}$$

where

$$\Lambda^\pm = \exp[i(\theta^a \mp i\eta^a)\sigma^a/2], \quad a = 1,2,3. \tag{54}$$

The parameters θ^a are rotation angles ranging $\langle 0, 2\pi\rangle$ each. The parameters η^a are rapidities[9], $-\infty < \eta < +\infty$. Recalling the relation (10), $\sigma^2\vec{\sigma}\sigma^2 = -\vec{\sigma}^T = -\vec{\sigma}^*$, it can be shown that

$$\sigma^2(\Lambda^\pm)^*\sigma^2 = \Lambda^\mp. \tag{55}$$

It means that Λ^+ and Λ^- are not equivalent representations. Therefore there might exist two different types of fermions of spin $1/2$.

Since $\Lambda^\pm$ is not unitary matrix hermite conjugated of $\Lambda^\pm$ is not inverse to $\Lambda^\pm$. Rather,

$$(\Lambda^\pm)^\dagger = (\Lambda^\mp)^{-1}. \tag{56}$$

Thus, to build Lorentz invariant objects out of the Weyl spinors of a single representation it will take a little bit more effort. If we define

$$\sigma^\mu = (\mathbf{I}^{(2)}, \sigma^a), \quad \bar{\sigma}^\mu = (\mathbf{I}^{(2)}, -\sigma^a), \tag{57}$$

we can prove that $(\xi^+)^\dagger\sigma^\mu\zeta^+$ and $(\xi^-)^\dagger\bar{\sigma}^\mu\zeta^-$ transform as contravariant four-vectors $(\frac{1}{2},\frac{1}{2})$

$$(\xi^+)^\dagger\sigma^\mu\zeta^+ \to \Lambda^\mu{}_\nu[(\xi^+)^\dagger\sigma^\nu\zeta^+], \quad (\xi^-)^\dagger\bar{\sigma}^\mu\zeta^- \to \Lambda^\mu{}_\nu[(\xi^-)^\dagger\bar{\sigma}^\nu\zeta^-], \tag{58}$$

where $\xi^\pm, \zeta^\pm$ represent Weyl spinors. Consequently,

$$(\Lambda^+)^\dagger\sigma^\mu\Lambda^+ = \Lambda^\mu{}_\nu\sigma^\nu, \quad (\Lambda^-)^\dagger\bar{\sigma}^\mu\Lambda^- = \Lambda^\mu{}_\nu\bar{\sigma}^\nu \tag{59}$$

The Lorentz invariant Lagrangians for $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ fermions are

$$L_f^+ = i(\xi^+)^\dagger\sigma^\mu\partial_\mu\xi^+, \quad L_f^- = i(\xi^-)^\dagger\bar{\sigma}^\mu\partial_\mu\xi^-, \tag{60}$$

respectively. Note that the explicit mass terms $m_+(\xi^+)^\dagger\xi^+$ and $m_-(\xi^-)^\dagger\xi^-$ are not Lorentz invariant; the Weyl spinors can represent massless fermions, only.

We know experimentally that gravity, electromagnetic and strong interactions preserve parity[10]. However, the parity transformation P changes representation (j_+, j_-) to (j_-, j_+). Particularly, $P : (\frac{1}{2},0) \leftrightarrow (0,\frac{1}{2})$ or

$$\Lambda_+ = \exp[i(\theta^a - i\eta^a)\sigma^a/2] \ \stackrel{P}{\longleftrightarrow} \ \Lambda_- = \exp[i(\theta^a + i\eta^a)\sigma^a/2]. \tag{61}$$

[9] If v^a are cartesian components of velocity then $v^a = \tanh\eta^a$ $(c = 1)$. Since η^a's range over non-compact intervals the Lorentz group is not a compact one. There is a theorem stating that non-compact groups have no unitary representations of finite dimension, except for representations in which the non-compact generators are represented trivially. Indeed, the matrices $\Lambda^\pm$ are not unitary. In this case, the non-compact generators are those responsible for boosts.

[10] Parity transforms $(t,\vec{x}) \to (t,-\vec{x})$. Consequently, under a parity transformation a velocity vector changes its sign while an angular momentum vector does not.

Obviously, the Lagrangians (60) are not parity invariant and a single Weyl spinor representation cannot be sufficient for building a parity invariant theory. Instead, we have to use both Weyl spinors, ξ^+ as well as ξ^-. The parity invariant fermion Lagrangian would then take the form

$$L_f^P = i(\xi^+)^\dagger \sigma^\mu \partial_\mu \xi^+ + i(\xi^-)^\dagger \bar{\sigma}^\mu \partial_\mu \xi^- - m[(\xi^-)^\dagger \xi^+ + (\xi^+)^\dagger \xi^-]. \tag{62}$$

Note that under parity $\partial_i \to -\partial_i$ which results in $P : \bar{\sigma}^\mu \partial_\mu \leftrightarrow \sigma^\mu \partial_\mu$.

In the case of parity invariant theories it is convenient to define objects which close under Lorentz as well as parity transformations. Such objects are *Dirac spinors*, or *bispinors*

$$\psi = \begin{pmatrix} \xi^- \\ \xi^+ \end{pmatrix}. \tag{63}$$

They have four complex components and under a parity transformation P the Dirac spinor changes as follows

$$\begin{pmatrix} \xi^- \\ \xi^+ \end{pmatrix} \xrightarrow{P} \begin{pmatrix} \xi^+ \\ \xi^- \end{pmatrix}. \tag{64}$$

Under the Lorentz group the bispinor transforms with the help of 4×4 complex matrix Λ_D

$$\psi \stackrel{SO(1,3)}{\longrightarrow} \Lambda_D \psi, \qquad \Lambda_D = \begin{pmatrix} \Lambda^- & 0 \\ 0 & \Lambda^+ \end{pmatrix}. \tag{65}$$

The inverse matrix to Λ_D is

$$\Lambda_D^{-1} = \begin{pmatrix} (\Lambda^-)^{-1} & 0 \\ 0 & (\Lambda^+)^{-1} \end{pmatrix} = \begin{pmatrix} (\Lambda^+)^\dagger & 0 \\ 0 & (\Lambda^-)^\dagger \end{pmatrix} = \gamma^0 \Lambda_D^\dagger \gamma^0, \tag{66}$$

where

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{I}^{(2)} \\ \mathbf{I}^{(2)} & 0 \end{pmatrix}.$$

Since $(\gamma^0)^2 = \mathbf{I}^{(4)}$ the object that transforms by Λ_D^{-1} from the right is a bispinor *Dirac conjugated* to ψ: $\bar{\psi} = \psi^\dagger \gamma^0$. Indeed,

$$\bar{\psi} = \psi^\dagger \gamma^0 \quad \to \quad \psi^\dagger \Lambda_D^\dagger \gamma^0 = \bar{\psi} \Lambda_D^{-1}. \tag{67}$$

Thus $\bar{\psi}\psi$ is a Lorentz invariant as oppose to $\psi^\dagger \psi$. Further, if we define the *Dirac matrices*

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \qquad \mu = 0, 1, 2, 3, \tag{68}$$

then with the help of (59) we get

$$\Lambda_D^{-1} \gamma^\mu \Lambda_D = \Lambda^\mu{}_\nu \gamma^\nu. \tag{69}$$

It implies that $\bar{\psi}\gamma^\mu\psi$ transforms as a Lorentz fourvector

$$\bar{\psi}\gamma^\mu\psi \quad \to \quad \Lambda^\mu{}_\nu (\bar{\psi}\gamma^\nu\psi). \tag{70}$$

Now, using ψ, the Lagrangian (62) can be rewritten as follows

$$L_f^P = i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi, \tag{71}$$

In parity invariant theories the spin 1/2 fermions have four[11] times the number of degrees of freedom of spin 0 particles described by the real $(0,0)$ representation (neutral scalars). As seen in (71) the parity invariant theories admit explicit mass terms if the mass is common to all four internal degrees of freedom.

[11] The factor four comes from two spin projections of the particle plus two spin projections of its antiparticle.

5.2 Charge Conjugation

Let us mention — for the sake of completeness — that there is another transformation which makes transition between the two sorts of the Weyl spinors. The operation is called *charge conjugation*. Let us define the charge conjugated spinors to the Weyl spinors $\xi^\pm$ as follows[12]

$$(\xi^-)^c \equiv i\sigma^2(\xi^-)^*, \quad (\xi^+)^c \equiv -i\sigma^2(\xi^+)^*. \tag{72}$$

Using (55) we can show that $(\xi^-)^c$ transforms as ξ^+ and $(\xi^+)^c$ transforms as ξ^-

$$(\xi^-)^c \to \Lambda_+(\xi^-)^c, \quad (\xi^+)^c \to \Lambda_-(\xi^+)^c,$$

and that

$$[(\xi^-)^c]^c = \xi^-, \quad [(\xi^+)^c]^c = \xi^+.$$

Obviously, charge conjugation is not a symmetry of the Lagrangians (60). Nevertheless, the Lagrangian (62) possesses the charge symmetry.

The operation of charge conjugation on a Dirac spinor ψ leads to the definition of a charge conjugated bispinor

$$\psi^c \equiv \begin{pmatrix} -i\sigma^2(\psi^+)^* \\ i\sigma^2(\psi^-)^* \end{pmatrix} = -i\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} (\psi^-)^* \\ (\psi^+)^* \end{pmatrix} = -i\gamma^2\psi^*. \tag{73}$$

When charge conjugation on the Dirac spinor is iterated twice it gives the identity

$$(\psi^c)^c = \psi.$$

In parity invariant theories charge conjugation changes particles to their antiparticles, and *vice versa*. On the other hand, the Lagrangians (60) are invariant only when both — parity transformation P and charge conjugation C — are performed; the so-called CP transformation. Then, antiparticles are described by CP-transformed Weyl spinors.

5.3 Chirality

The bispinor space is made up of two Lorentz invariant subspaces, each hosting one sort of the Weyl spinors. Dirac spinors can be projected on these subspaces with projection operators built from the Dirac gamma matrices

$$\psi_\ell \equiv \frac{1}{2}(1-\gamma^5)\psi = \begin{pmatrix} \xi^- \\ 0 \end{pmatrix}, \quad \psi_r \equiv \frac{1}{2}(1+\gamma^5)\psi = \begin{pmatrix} 0 \\ \xi^+ \end{pmatrix}, \tag{74}$$

where $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. We have used ℓ(eft) and r(ight) for the subscripts of projected bispinors to point out that they are connected to each other through the operation of parity which in space transforms "right things" to "left things" and *vice versa*. If in a theory of fermions parity is not a symmetry[13] the theory has to be formulated explicitly in terms of left and right bispinors $\psi_{\ell,r}$

[12] The factors $\pm i$ are introduced so that the twofold application of the charge conjugation equals to the identity transformation. The charge conjugation does not transform space-time coordinates.

[13] This happens when left and right fermions are subject to different interactions. In addition, if the interactions of left and right fermions are tied to different symmetry groups then the left and right fermions transform differently.

(or in terms of Weyl spinors). Then the left and right bispinors each represent different fermions. The Lagrangians L_f^+ and L_f^- in (60) can be rewritten in terms of the left and right bispinors

$$L_f^+ = i\bar{\psi}_r\gamma^\mu\partial_\mu\psi_r, \quad L_f^- = i\bar{\psi}_\ell\gamma^\mu\partial_\mu\psi_\ell. \tag{75}$$

The left and right fermions as well as their Lagrangians (theory) are called *chiral.* In CP-invariant theories ψ_ℓ^c describes an antiparticle to ψ_r, and ψ_r^c is an antiparticle to[14] ψ_ℓ.

In previous sections we were building an $SU(2)_L \times SU(2)_R$ invariant theory where $SU(2)_L \times SU(2)_R$ was an internal symmetry which puts restrictions on the form of Lagrangian. Fields were grouped into multiplets members of which were sharing the same interactions and masses. In particular, fermion fields were forming $SU(2)_L$ and $SU(2)_R$ doublets (see the eq. (39))

$$SU(2)_L \times SU(2)_R: \ \Psi_I' = U_I\Psi_I, \quad \Psi_I \equiv \begin{pmatrix} u_I \\ d_I \end{pmatrix}, \quad I = L, R,$$

where $U_I \in SU(2)_I$, and u_I and d_I are Dirac spinors. Note that our $SU(2)_L \times SU(2)_R$ theory is parity invariant.

However, if parity is not a symmetry of Nature we have to reconsider our Lagrangians and formulate them in terms of chiral fermions. For example, in the $SU(2)_L \times SU(2)_R$ theory there might exist one $SU(2)_L$ doublet formed by left fermions and one $SU(2)_R$ doublet formed by right fermions. Strictly speaking, following our previous notation we should write

$$\Psi_L = \begin{pmatrix} u_{L\ell} \\ d_{L\ell} \end{pmatrix}, \quad \Psi_R = \begin{pmatrix} u_{Rr} \\ d_{Rr} \end{pmatrix}, \tag{76}$$

where capital letters express transformation properties under the $SU(2)_L\times SU(2)_R$ group and small letters denote chirality of the fermions. Obviously, there is a great danger of misunderstanding in this situation. There are two different $SU(2) \times SU(2)$ groups in play and, in the literature, the same subscripts, L and R, are used to distinguish components of the direct products in both cases. While these particular letters are meaningful for distinguishing representations of the Lorentz group, in the case of $SU(2)_L \times SU(2)_R$ transformations they have no particular meaning and they might have been chosen otherwise. In the latter case the choice of L and R is motivated by anticipated connection of the individual $SU(2)$ fermion doublets to left and right fermions.

Let us check[15] the consequences of (76) for our theory. First of all, the free fermion Lagrangian cannot have the mass terms since $\bar{\psi}_\ell\psi_\ell = \bar{\psi}_r\psi_r = 0$. Thus (40) is modified to

$$L_f = \bar{\Psi}_L i\not{\partial}\,\Psi_L + \bar{\Psi}_R i\not{\partial}\,\Psi_R. \tag{77}$$

The interactions of fermions to the vector fields will not get modified. Thus, following (41),

$$L_{fV} = -g_L^{fV}\bar{\Psi}_L\gamma^\mu\mathcal{L}_\mu\Psi_L - g_R^{fV}\bar{\Psi}_R\gamma^\mu\mathcal{R}_\mu\Psi_R. \tag{78}$$

Eventually, there are interactions between fermions and the scalar fields. Again, nothing will change in (51), so

$$L_{fH} = c_f(\bar{\Psi}_L M\Psi_R + \bar{\Psi}_R M^\dagger\Psi_L). \tag{79}$$

[14]For example, if neutrinos in the SM were massless then the antiparticle to the left neutrino would be the right antineutrino. The existence of right neutrinos and left antineutrinos would not be necessary.

[15]For this purpose it would be handy to summarize some basic relations for the projection matrices $\mathcal{P}_{r,\ell} \equiv (1 \pm \gamma^5)/2$: $\mathcal{P}_r + \mathcal{P}_\ell = \mathbf{I}$, $\mathcal{P}_r\mathcal{P}_\ell = 0$, $\mathcal{P}_{r,\ell}^2 = \mathcal{P}_{r,\ell}$, $\psi_{r,\ell} = \mathcal{P}_{r,\ell}\psi$, $\bar{\psi}_{r,\ell} = \bar{\psi}\mathcal{P}_{\ell,r}$, $(\mathcal{P}_{r,\ell})^\dagger = \mathcal{P}_{r,\ell}$, $\mathcal{P}_{r,\ell}\gamma^\mu = \gamma^\mu\mathcal{P}_{\ell,r}$, $\mathcal{P}_r\gamma^5 = \mathcal{P}_r$, $\mathcal{P}_\ell\gamma^5 = -\mathcal{P}_\ell$, $\bar{\mathcal{P}}_{r,\ell} \equiv \gamma^0\mathcal{P}_{r,\ell}^\dagger\gamma^0 = \mathcal{P}_{\ell,r}$.

Note that this Lagrangian couples fermions from different doublets, and thus with different chirality. When SSB is taken into account the L_{ff} part of (79) has a form

$$L_{ff} = vc_f(\bar{\Psi}_L\Psi_R + \bar{\Psi}_R\Psi_L) = vc_f(\bar{u}_{L\ell}u_{Rr} + \bar{u}_{Rr}u_{L\ell}) + vc_f(\bar{d}_{Rr}d_{L\ell} + \bar{d}_{L\ell}d_{Rr}) = vc_f\bar{.} \tag{80}$$

The simplest interpretation of L_{ff} is that $u_{L\ell}$ *vs* u_{Rr} and $d_{L\ell}$ *vs* d_{Rr} are not different particles. Rather they are left and right projections of the same fermion, u and d, respectively:

$$u_{L\ell,Rr} = \mathcal{P}_{\ell,r}u, \quad d_{L\ell,Rr} = \mathcal{P}_{\ell,r}d$$

Note that in this case the notation with two subscripts becomes redundant. We can survive with one index denoting to which $SU(2)$ doublet the fermion belongs as well as what is the fermion's chirality. Thus we can replace $u_{L\ell,Rr} \to u_{L,R}$, $d_{L\ell,Rr} \to d_{L,R}$. The most important consequence of this interpretation is that through SSB chiral fermions can receive masses. In addition, the $SU(2)_L \times SU(2)_R$ symmetry of L_{fH}, results in both fermions, u and d, having the same masses

$$m_u = m_d = vc_f.$$

6 Gauging the $SU(2)_L \times SU(2)_R$ Lagrangian

Let us say we wish[16] to promote the global $SU(2)_L \times SU(2)_R$ symmetry of the theory to the local one. Locality of a transformation means that it depends on the space-time position. The dependency translates into group parameters being functions of space-time coordinates x. Then, under local $G = SU(2)_L \times SU(2)_R$, the matrix M of scalar fields in (15) transforms as

$$M \xrightarrow{G} M' = U_L(x)MU_R^\dagger(x), \quad U_{L,R}(x) = \exp[i\alpha^a_{L,R}(x)\tau^a]. \tag{81}$$

Similarly, the local transformations for $SU(2)$ fermion doublets (39) read

$$\Psi_{L,R} \xrightarrow{G} \Psi'_{L,R} = U_{L,R}(x)\Psi_{L,R}. \tag{82}$$

Due to the presence of terms with partial derivatives neither the Higgs Lagrangian L_H (23) nor the fermion Lagrangian L_f (40) are invariant under the local transformations. Note that in (40) the requirement of the local symmetry does not exclude the presence of explicit mass terms for fermions. Also the scalar boson self-interactions in (23) are locally symmetric. The interaction Lagrangian (51) which couples fermions to scalars is locally invariant.

If the local transformation of the vector fields $\{\mathcal{L}^a_\mu\}^3_{a=1}$ and $\{\mathcal{R}^a_\mu\}^3_{a=1}$ possessed the following form

$$\mathcal{L}'_\mu = U_L(x)\mathcal{L}_\mu U_L^\dagger(x), \quad \mathcal{R}'_\mu = U_R(x)\mathcal{R}_\mu U_R^\dagger(x)$$

the field-strength tensors would not transform as in (35) and (36) and the vector field Lagrangian (37) would not be locally invariant. This would not be true about the explicit mass terms in L_V, though. They would be locally invariant. The interactions of the vector fields to fermions (41) would survive unchanged. This does not apply to all interaction terms L_{VH} in (43).

The well-known recipe how to achieve the local invariance of all the Lagrangians mentioned above is to change the transformation pattern for the vector bosons to

$$\mathcal{L}'_\mu = U_L\mathcal{L}_\mu U_L^\dagger + \frac{i}{g_L}(\partial_\mu U_L)U_L^\dagger, \tag{83}$$

$$\mathcal{R}'_\mu = U_R\mathcal{R}_\mu U_R^\dagger + \frac{i}{g_R}(\partial_\mu U_R)U_R^\dagger, \tag{84}$$

[16] At this point we are not going to discuss the motivation of this requirement.

where $g_{L,R}$ are the vector-boson self-interaction coupling constants introduced in (33) and (34). Then

$$\mathcal{L}'_{\mu\nu} = U_L(x)\mathcal{L}_{\mu\nu}U_L^\dagger(x), \quad \mathcal{R}'_{\mu\nu} = U_R(x)\mathcal{R}_{\mu\nu}U_R^\dagger(x) \tag{85}$$

and the $SU(2)_L \times SU(2)_R$ locally invariant vector boson Lagrangian is

$$L_V = -\frac{1}{4}\mathrm{Tr}(\mathcal{L}_{\mu\nu}\mathcal{L}^{\mu\nu}) - \frac{1}{4}\mathrm{Tr}(\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}). \tag{86}$$

Note that the explicit mass terms present in (37) must have been given up in (86) since they were not invariant under (83), (84). Thus the local invariance prevents from assigning an explicit mass to vector bosons.

In addition, the transformations (83), (84) spoil the invariance of L_{fV} in (41) and L_{VH} in (43). However, if we set

$$g_L^{fV} = g_L, \quad g_R^{fV} = g_R \tag{87}$$

the sum $L_f + L_{fV}$

$$\begin{aligned} L_f^{gauged} \equiv L_f + L_{fV} &= \bar{\Psi}_L(i\,\partial\!\!\!/ - m_L)\Psi_L - g_L\bar{\Psi}_L\gamma^\mu\mathcal{L}_\mu\Psi_L \\ &\quad + \bar{\Psi}_R(i\,\partial\!\!\!/ - m_R)\Psi_R - g_R\bar{\Psi}_R\gamma^\mu\mathcal{R}_\mu\Psi_R \end{aligned} \tag{88}$$

becomes locally invariant. If the doublets Ψ_L and Ψ_R are made up of chiral fermions, left ones and right ones, respectively, then the Lorentz invariance requires $m_L = m_R = 0$. The mass term $m(\bar{\Psi}_R\Psi_L + \bar{\Psi}_R\Psi_L)$ is not $SU(2)_L \times SU(2)_R$ invariant and thus not allowed, either.

As far as the Higgs sector is concerned, if

$$g_{L,R}^{VH} = ig_{L,R}, \quad g_{LL,RR}^{VH} = g_{L,R}^2, \quad g_{RL}^{VH} = -g_R g_L \tag{89}$$

then the sum $L_H + L_{VH}$

$$\begin{aligned} L_H^{gauged} \equiv L_H + L_{VH} &= \frac{1}{4}\mathrm{Tr}[(\partial_\mu M^\dagger - ig_L M^\dagger\mathcal{L}_\mu + ig_R\mathcal{R}_\mu M^\dagger)(\partial^\mu M + ig_L\mathcal{L}^\mu M - ig_R M\mathcal{R}^\mu)] \\ &\quad + \frac{\mu^2}{4}\mathrm{Tr}(M^\dagger M) - \frac{\lambda}{16}[\mathrm{Tr}(M^\dagger M)]^2 \end{aligned} \tag{90}$$

also becomes locally invariant.

Therefore, the additional piece of the recipe is to replace in L_H and L_f the partial derivatives of $M(x)$ and $\Psi_{L,R}(x)$ with the so-called *covariant derivatives*

$$D_\mu M = \partial_\mu M + ig_L\mathcal{L}_\mu M - ig_R M\mathcal{R}_\mu, \tag{91}$$

$$D_\mu\Psi_L = (\partial_\mu + ig_L\mathcal{L}_\mu)\Psi_L, \tag{92}$$

$$D_\mu\Psi_R = (\partial_\mu + ig_R\mathcal{R}_\mu)\Psi_R. \tag{93}$$

Then, the Lagrangians L_f^{gauged} and L_H^{gauged} can be rewritten in terms of the covariant derivatives

$$L_f^{gauged} = i(\bar{\Psi}_L D\!\!\!\!/\,\Psi_L + \bar{\Psi}_R D\!\!\!\!/\,\Psi_R). \tag{94}$$

$$L_H^{gauged} = \frac{1}{4}\mathrm{Tr}(D_\mu M D^\mu M^\dagger) - V(M). \tag{95}$$

Since the covariant derivatives transform in the following way

$$(D_\mu M)' = U_L(D_\mu M)U_R^\dagger, \quad (D_\mu\Psi_L)' = U_L(D_\mu\Psi_L), \quad (D_\mu\Psi_R)' = U_R(D_\mu\Psi_R), \tag{96}$$

it is easy to check that both gauged Lagrangians are locally $SU(2)_L \times SU(2)_R$ invariant.

Note that the requirement of the local invariance led to the reduction (unification) of coupling constants in the theory, as is seen in the eqs. (87) and (89). Seven free parameters were reduced down to just two, g_L and g_R. In addition, the symmetry requirements eliminated some mass terms which further reduced the number of free parameters in the theory.

Substituting (89) into (46) through (50) we obtain the following eigenvalues of $\mathcal{M}^2$

$$\mu_1^2 = \left(\frac{vG}{2}\right)^2, \quad \mu_2^2 = 0, \tag{97}$$

where $G = \sqrt{g_L^2 + g_R^2}$. For the mixing angles we get

$$\sin\theta = \frac{g_R}{G}, \qquad \cos\theta = \frac{g_L}{G}. \tag{98}$$

Then L_{Vmass} in terms of the mass eigenstates $\vec{\mathcal{T}}_\mu$ and $\vec{\mathcal{U}}_\mu$ is

$$\begin{aligned} L_{Vmass} \;=\; & \frac{1}{2}(\mathcal{T}^3_\mu, \mathcal{U}^3_\mu)\begin{pmatrix} (vG/2)^2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \mathcal{T}^{3\mu} \\ \mathcal{U}^{3\mu} \end{pmatrix} \\ & +\frac{1}{2}(\mathcal{T}^-_\mu, \mathcal{U}^-_\mu)\begin{pmatrix} (vG/2)^2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \mathcal{T}^{+\mu} \\ \mathcal{U}^{+\mu} \end{pmatrix} \\ & +\frac{1}{2}(\mathcal{T}^+_\mu, \mathcal{U}^+_\mu)\begin{pmatrix} (vG/2)^2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \mathcal{T}^{-\mu} \\ \mathcal{U}^{-\mu} \end{pmatrix}. \end{aligned} \tag{99}$$

This suggests that there are three vector bosons $\mathcal{T}^3, \mathcal{T}^+, \mathcal{T}^-$ of the same mass

$$m_{\mathcal{T}^3} = m_{\mathcal{T}^+} = m_{\mathcal{T}^-} = \frac{vG}{2}, \tag{100}$$

along with three massless vector bosons $\mathcal{U}^3, \mathcal{U}^+, \mathcal{U}^-$.

7 Lagrangian based on the $SU(2)_L \times U(1)_{R3}$ symmetry

Recall that $SU(2)_L \times U(1)_{R3}$ is a subgroup of $SU(2)_L \times SU(2)_R$. It transforms M in the following way

$$M' = U_L M U_{R3}^\dagger, \quad U_L = \exp(i\vec{\alpha}\vec{\tau}), \;\; U_{R3} = \exp(i\beta\tau^3). \tag{101}$$

Under $SU(2)_L \times U(1)_{R3}$ group not only M but also the objects Φ and $\tilde{\Phi}$ of $M = \sqrt{2}(\tilde{\Phi}, \Phi)$ have well defined transformation properties

$$SU(2)_L : \qquad \Phi \to U_L\Phi, \;\; \tilde{\Phi} \to U_L\tilde{\Phi}, \tag{102}$$

$$U(1)_{R3} : \qquad \Phi \to e^{i\beta/2}\Phi, \;\; \tilde{\Phi} \to e^{-i\beta/2}\tilde{\Phi}. \tag{103}$$

The Lagrangian L_H (23) can be expressed in terms of Φ

$$L_\Phi \equiv (\partial_\mu\Phi)^\dagger(\partial^\mu\Phi) + \underbrace{\mu^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2}_{-V(\Phi)} = L_H(M(\Phi)) \tag{104}$$

and obviously it is $SU(2)_L \times U(1)_{R3}$ globally invariant[17]. The minimum of the potential $V(\Phi)$ is at Φ's satisfying the condition

$$\Phi^\dagger \Phi = \frac{\mu^2}{2\lambda} = \frac{v^2}{2}. \tag{105}$$

This condition is invariant under $SU(2)_L \times U(1)_{R3}$ transformations of Φ.

The choice of the physical vacuum that corresponds to (26) is

$$\Phi_{vac} = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}. \tag{106}$$

The vacuum is less symmetric than the Lagrangian (104). Namely, it is invariant under $U(1)_{V3}$ transformations

$$\Phi'_{vac} = \exp(i\beta\tau^3)\exp(i\beta/2)\Phi_{vac} = \begin{pmatrix} \exp(i\beta) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} = \Phi_{vac}. \tag{107}$$

Under $U(1)_{V3}$ the scalar complex doubet Φ transforms in the following way

$$\Phi' = \exp[i\beta(\tau^3 + \frac{1}{2}\mathbf{I}^{(2)})] \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} e^{i\beta}\varphi_1 \\ \varphi_2 \end{pmatrix}, \tag{108}$$

$$\tilde{\Phi}' = \exp[i\beta(\tau^3 - \frac{1}{2}\mathbf{I}^{(2)})] \begin{pmatrix} \varphi_2^* \\ -\varphi_1^* \end{pmatrix} = \begin{pmatrix} \varphi_2^* \\ -e^{-i\beta}\varphi_1^* \end{pmatrix}, \tag{109}$$

where we have used 11 and (12) for parameterization of Φ and $\tilde{\Phi}$. The relations (108) and (109) can be summarized in a single compact relation valid for each complex scalar field φ_i

$$\varphi' = \exp(i\beta Q)\varphi, \tag{110}$$

where the parameter $Q \equiv T^3 + Y$ is introduced in such a way that

φ	T^3	Y	Q
φ_1	$+1/2$	$+1/2$	$+1$
φ_2	$-1/2$	$+1/2$	0
φ_2^*	$+1/2$	$-1/2$	0
φ_1^*	$-1/2$	$-1/2$	-1

(111)

Obviously, the T^3 parameter is connected to the $SU(2)_L$ transfomation properties of the field φ and Y is related to its $U(1)_{R3}$ properties. The parameters are called *quantum numbers.*

The fields of Φ can be reparameterized in the usual way reflecting the choice of the physical vacuum (106)

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \pi^2(x) + i\pi^1(x) \\ v + h(x) - i\pi^3(x) \end{pmatrix}. \tag{112}$$

[17] It is also $SU(2)_L \times SU(2)_R$ invariant. One just has to be carreful about proper applying of $SU(2)_R$ transformation relations if the Lagrangian is expressed in terms of Φ or, which would be a better approach, write the Lagrangian in terms of M.

Substituting (112) into (104) we again end up with three massless scalar fields π^a and one massive scalar field h of the mass $m_h = \sqrt{2}\mu$.

To promote the global $SU(2)_L \times U(1)_{R3}$ symmetry of L_H to the local one appropriate gauge fields have to be introduced: the $SU(2)_L$ triplet $\{W^a_\mu\}^3_{a=1}$ and the $U(1)_{R3}$ singlet B_μ with the following transformation properties

$$W'_\mu = U_L W_\mu U_L^\dagger + \frac{i}{g}(\partial_\mu U_L)U_L^\dagger, \qquad W_\mu \equiv W^a_\mu \tau^a, \tag{113}$$

$$\mathcal{B}'_\mu = U_{R3}\mathcal{B}_\mu U_{R3}^\dagger + \frac{i}{g'}(\partial_\mu U_{R3})U_{R3}^\dagger, \qquad \mathcal{B}_\mu \equiv B_\mu \tau^3, \tag{114}$$

where g and g' denote gauge couplings. Then, the covariant derivative

$$D_\mu M = \partial_\mu M + igW_\mu M - ig' M\mathcal{B}_\mu \tag{115}$$

transforms as follows

$$(D_\mu M)' = U_L(D_\mu M)U_{R3}^\dagger. \tag{116}$$

The Lagrangian L_H (23) will become locally $SU(2)_L \times U(1)_{R3}$ symmetric after replacing $\partial_\mu M \to D_\mu M$.

Now the considerations of the previous paragraph will be carried out in terms of Φ. First, equation (114) along with (101) implies

$$B'_\mu = B_\mu - \frac{1}{g'}\partial_\mu \beta. \tag{117}$$

The covariant derivatives of Φ and Φ' are obtained from (115) when we plug in $M = \sqrt{2}(\tilde{\Phi}, \Phi)$ and $\mathcal{B}_\mu = B_\mu \tau^3$

$$D_\mu \Phi = (\partial_\mu + igW_\mu + \frac{i}{2}g' B_\mu)\Phi, \tag{118}$$

$$D_\mu \tilde{\Phi} = (\partial_\mu + igW_\mu - \frac{i}{2}g' B_\mu)\tilde{\Phi}. \tag{119}$$

They transform as follows

$$(D_\mu \Phi)' = U_L e^{i\beta/2}(D_\mu \Phi), \quad (D_\mu \tilde{\Phi})' = U_L e^{-i\beta/2}(D_\mu \tilde{\Phi}). \tag{120}$$

The Lagrangian L_Φ (104) will become locally $SU(2)_L \times U(1)_{R3}$ symmetric after replacing $\partial_\mu \Phi \to D_\mu \Phi$

$$L_\Phi = (D_\mu \Phi)^\dagger (D^\mu \Phi) + \mu^2 \Phi^\dagger \Phi - \lambda(\Phi^\dagger \Phi)^2. \tag{121}$$

7.1 $SU(2)_L \times U(1)_{R3}$ fermions

Let us introduce fermions: $SU(2)_L$ doublet Ψ_L and $U(1)_{R3}$ singlet ψ_R. Under $SU(2)_L \times U(1)_{R3}$ group they transform as follows[18]

$$SU(2)_L: \qquad \Psi_L \to \exp(i\vec{\alpha}\vec{\tau})\Psi_L, \quad \psi_R \to \psi_R, \tag{122}$$

$$U(1)_{R3}: \qquad \Psi_L \to \exp(i\beta Y \mathbf{I}^{(2)})\Psi_L, \quad \psi_R \to \exp(i\beta Y)\psi_R, \tag{123}$$

[18]Note that Φ as well as Ψ_L being the $SU(2)_L$ doublets had to be introduced as $U(1)_{R3}$ doublets at the same time. It is not the case for the fermion fields ψ_R, though; they can be introduced as $U(1)_{R3}$ singlets, thus allowing various values of Y for individual ψ_R.

where Y is the $U(1)_{R3}$ quantum number of the fermion field under transformation. The $U(1)_{V3}$ ransformations read

$$U(1)_{V3}: \quad \Psi_L \to \exp[i\beta(\tau^3 + Y\mathbf{I}^{(2)})]\Psi_L, \quad \psi_R \to \exp(i\beta Y)\psi_R. \tag{124}$$

It can be written in a compact way $\psi \to \exp(i\beta Q)\psi$ where $Q = T^3 + Y$ if

$$T^3(u_L) = \frac{1}{2}, \quad T^3(d_L) = -\frac{1}{2}, \quad T^3(\psi_R) = 0. \tag{125}$$

The covariant derivatives of the fermion fields are

$$\begin{aligned} D_\mu \Psi_L &= (\partial_\mu + igW^a_\mu \tau^a + ig'Y\mathbf{I}^{(2)}B_\mu)\Psi_L, && (126)\\ D_\mu \psi_R &= (\partial_\mu + ig'YB_\mu)\psi_R. && (127) \end{aligned}$$

Then the gauge invariant fermion kinetic terms can be written in the form

$$L_{\mathcal{F}} = i(\bar{\Psi}_L \not{D}\Psi_L + \bar{\psi}_R \not{D}\psi_R). \tag{128}$$

Again, through the gauging of the kinetic Lagrangian also the interactions of fermions to the gauge bosons have been introduced.

Respecting the global $SU(2)_L \times U(1)_{R3}$ symmetry the fermionic part of the Lagrangian responsible for interactions to chiral fermions becomes

$$L_{\Phi f} = \sqrt{2}c_u(\bar{\Psi}_L\tilde{\Phi}u_R + \bar{u}_R\tilde{\Phi}^\dagger\Psi_L) + \sqrt{2}c_d(\bar{\Psi}_L\Phi d_R + \bar{d}_R\Phi^\dagger\Psi_L), \tag{129}$$

where c_u, c_d are coupling constants. Note that in (129) $c_u = c_d$ restores the $SU(2)_R$ symmetry of (51).

8 Appendices

A Diagonalization of a 2×2 Symmetric Real Matrix

A 2×2 symmetric real matrix:

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \qquad a, b, c \in \mathcal{R} \tag{130}$$

It can be understood as a matrix representation of $\hat{M}$ in the orthogonal basis $\{e_1, e_2\}$

$$M_{ij} = (e_i, \hat{M} e_j)$$

where the round brackets denote the scalar product.
eigenvalues of M:

$$m_{1,2} = \frac{1}{2}(a+c) \pm \frac{1}{2}\sqrt{(a-c)^2 + 4b^2} \tag{131}$$

eigenvectors[19] of M:

$$v_1 = N_1 \begin{pmatrix} 1 \\ \frac{m_1 - a}{b} \end{pmatrix} = \begin{pmatrix} (e_1, v_1) \\ (e_2, v_1) \end{pmatrix}, \qquad v_2 = N_2 \begin{pmatrix} 1 \\ \frac{m_2 - a}{b} \end{pmatrix} = \begin{pmatrix} (e_1, v_2) \\ (e_2, v_2) \end{pmatrix} \tag{132}$$

where N_1, N_2 are normalization costants. From $(v_1, v_1) = (v_2, v_2) = 1$

$$N_1 = \pm \frac{|b|}{\sqrt{b^2 + (m_1 - a)^2}}, \qquad N_2 = \pm \frac{|b|}{\sqrt{b^2 + (m_2 - a)^2}} \tag{133}$$

where the choice of signs in N_1 and N_2 are independent of each other. Some auxiliary relations

$$m_1 + m_2 = a + c, \qquad m_1 m_2 = ac - b^2 \tag{134}$$

$$(m_1 - a)(m_2 - a) = -b^2 \tag{135}$$

The eigenvectors are orthogonal to each other, $(v_1, v_2) = 0$.

The diagonalized M:

$$M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \tag{136}$$

which is a matrix representation of $\hat{M}$ in the basis $\{v_1, v_2\}$,

$$M_{ij} = (v_i, \hat{M} v_j) = m_j \delta_{ij}$$

Obviously, in this basis

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

[19] In this Appendix none of the relations below this point depend on which of the eigenvalues in (131) is called m_1 and which one is called m_2.

Transformation of the basis $\{v_1, v_2\}$ to $\{e_1, e_2\}$:

$$e_i = O_{ij} v_j, \qquad O^T O = \mathbf{I} \tag{137}$$

Then

$$O_{ij} = (e_i, v_j)$$

Coordinates of a vector x transform as follows

$$(e_i, x) = O_{ij}(v_j, x) \tag{138}$$

or in the matrix form

$$\left(\begin{array}{c} (e_1, x) \\ (e_2, x) \end{array} \right) = \left(\begin{array}{cc} (e_1, v_1) & (e_1, v_2) \\ (e_2, v_1) & (e_2, v_2) \end{array} \right) \left(\begin{array}{c} (v_1, x) \\ (v_2, x) \end{array} \right) \tag{139}$$

Comparing (139) with (132) we obtain[20]

$$O = (v_1, v_2) = \left(\begin{array}{cc} N_1 & N_2 \\ N_1 \frac{m_1 - a}{b} & N_2 \frac{m_2 - a}{b} \end{array} \right) \tag{140}$$

where v_i are columns of the normalized eigenvectors (132). Since the transformation matrix O is orthogonal it can be parameterized as follows

$$O = \left(\begin{array}{cc} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{array} \right) \tag{141}$$

Comparing (140) with (141) we get

$$N_1 = N_2 \frac{m_2 - a}{b} = \cos\theta$$

which fixes the relative signs in the normalization constants. If we choose

$$\sin\theta = N_2 = \frac{\pm |b|}{\sqrt{b^2 + (m_2 - a)^2}} \tag{142}$$

then

$$\cos\theta = \frac{m_2 - a}{b} \sin\theta = \pm \frac{m_2 - a}{\sqrt{b^2 + (m_2 - a)^2}} \frac{|b|}{b} = \pm \frac{b}{\sqrt{b^2 + (m_1 - a)^2}}, \tag{143}$$

$$\tan\theta = \frac{b}{m_2 - a} = \frac{a - m_1}{b}. \tag{144}$$

[20] Here the round brackets do not represent the scalar product. They delimit elements of the transformation matrix.