Symmetries of Field Theory Lagrangians

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1 Some important Lie groups

$U(n)$:
The defining (*fundamental*) representation of the $U(1)$ group consists of all unitary $n \times n$ matrices. They must satisfy the unitarity requirement

$$UU^{\dagger} = 1. \tag{1}$$

The $U(1)$ group elements are complex numbers of unit modulo, i.e. of the form $\exp(i\alpha)$.

$SU(n)$:
The same as $U(n)$ plus the condition

$$\mathrm{det} U = 1. \tag{2}$$

Any unitary matrix can be written as

$$U = \exp(iH), \tag{3}$$

where H is a Hermitean matrix. The condition (2) implies that H is traceless. There are $N^2 - 1$ linearly independent $N \times N$ Hermitean traceless matrices T^a. Thus

$$U = \exp(i\alpha^a T^a), \qquad \alpha^a \in \mathcal{R}, a = 1, \ldots, N^2 - 1, \tag{4}$$

where T^a are so-called *generators*. Their commutator $[T^a, T^b]$ is antihermitean and traceless. Hence it can be expressed as a linear combination of T^a's: $[T^a, T^b] = if^{abc}T^c$. The commutation relation defines the algebra of the group. The coefficients f^{abc} of the linear combination are called the *structure constants* of the group. In the case of $SU(2)$ the commutator is

$$[T^a, T^b] = i\varepsilon^{abc}T^c, \tag{5}$$

where ε^{abc} is the Levi-Civita total antisymmetric tensor, and $T^a \equiv \tau^a = \sigma^a/2$, where $\sigma^1, \sigma^2, \sigma^3$ are the Pauli matrices.

The fundamental representation of $SU(N)$ acts on an N-dimensional vector

$$\xi'^i = U^i_{\ j}\xi^j, \qquad i, j = 1, \ldots, N. \tag{6}$$

There is a non-trivial N dimensional representation that is complex conjugate to the $SU(N)$ fundamental representation[1]. The complex conjugate representation can be obtained from (6)

$$(\xi'^i)^* = (U^i_{\ j})^*(\xi^j)^* = (U^{\dagger})^j_{\ i}(\xi^j)^*. \tag{7}$$

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[1] In general, given a matrix representation of a group the complex conjugate matrices also comprise a representation of the same group, *the complex conjugate representation.*

It is convenient to define an object ξ_i which transforms in the same way as the complex conjugate vector $(\xi^i)^*$. Then

$$\xi'_i = \xi_j (U^\dagger)^j_{\ i}. \tag{8}$$

We used upper and lower indices to distinguish vectors transformed by different N-dimensional representations of the $SU(N)$ group. Sometimes the upper index vector is called *covariant* and the lower index vector *contravariant.*

Let us consider a tensor Ξ^i_j which transforms as a product of a covariant and a contravariant vectors

$$\Xi'^i_j = U^i_{\ k} (U^\dagger)^l_{\ j} \Xi^k_l, \tag{9}$$

or

$$\Xi' = U \Xi U^\dagger. \tag{10}$$

If the tensor is traceless it defines *the adjoint representation* of $SU(N)$ group. If the matrix Ξ is hermitean[2] it can be written as a linear combination of the generators T^a, $\Xi = \omega^a T^a$.

The $SU(2)$ group is locally isomorphic to $SO(3)$ which is the group of three-dimensional rotations. Indeed, any 2×2 hermitean traceless matrix X can be expressed as a linear combination of the Pauli matrices, $X = x^a \sigma^a = \vec{x}\vec{\sigma}$. Let us note that $(\mathrm{Tr} X^2)/2 = (\vec{x})^2$. The $SU(2)$ transformations do not spoil hermitivity and tracelessness of X, thus $X' = UXU^\dagger = \vec{x}'\vec{\sigma}$. Finally, since $\vec{x}'^2 = (\mathrm{Tr} X'^2)/2 = \mathrm{Tr}(UXU^\dagger UXU^\dagger)/2 = (\mathrm{Tr} X^2)/2 = \vec{x}^2$ the $SU(2)$ transformations rotate the vector $\vec{x}$ to the vector $\vec{x}' = R\vec{x}, R \in SO(3)$. Thus we can associate a rotation with any given $U \in SU(2)$. Since U and $-U$ give the same rotation the group $SU(2)$ double covers $SO(3)$.

2 Global internal symmetries of free-fermion Lagrangians

A free electron:

$$\mathcal{L} = \bar{\psi} i \partial\!\!\!/ \psi - m_e \bar{\psi}\psi. \tag{11}$$

This Lagrangian is invariant under $U(1)$ transformations

$$\psi \to \exp(i\alpha)\psi, \qquad \bar{\psi} \to \exp(-i\alpha)\bar{\psi}. \tag{12}$$

However, the Lagrangian can be rewritten as

$$\mathcal{L} = \bar{\psi}_L i \partial\!\!\!/ \psi_L + \bar{\psi}_R i \partial\!\!\!/ \psi_R - m_e (\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L), \tag{13}$$

where $\psi_{L,R} = (1 \mp \gamma_5)/2$. In the case of massless electrons, $m_e = 0$, the Lagrangian acquires even higher symmetry, namely $U(1)_L \times U(1)_R$, under which the fields transform as follows

$$\psi_{L,R} \to \exp(i\alpha_{L,R})\psi_{L,R}, \qquad \exp(i\alpha_{L,R}) \in U(1)_{L,R}. \tag{14}$$

The symmetry in which "left" and "right" fields of the same particle can transform independently is called *chiral.* The group $U(1)$ of (12) is a "diagonal" subgroup of $U(1)_L \times U(1)_R$; it is obtained by putting $\alpha_L = \alpha_R$ in (14).

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[2] The adjoint representation of the $SU(N)$ transformations preserve (anti)hermiticity.

A free electron and a free neutrino:

$$\mathcal{L} = \bar{\nu} i \partial\!\!\!/ \nu + \bar{e} i \partial\!\!\!/ e - m_\nu \bar{\nu}\nu - m_e \bar{e} e. \tag{15}$$

This Lagrangian is invariant under $U(1)_\nu \times U(1)_e$:

$$\nu \rightarrow \exp(i\alpha_\nu)\nu, \quad \exp(i\alpha_\nu) \in U(1)_\nu, \tag{16}$$

$$e \rightarrow \exp(i\alpha_e)e, \quad \exp(i\alpha_e) \in U(1)_e. \tag{17}$$

If $m = m_\nu = m_e$ the both particles would behave in the same manner. Thus the whole system should have higher symmetry reflecting the equivalence of the neutrino and the electron. Assuming equal masses the Lagrangian (15) can be rewritten in a more compact form

$$\mathcal{L} = \bar{\Psi} i (\mathcal{I}_2 \otimes \partial\!\!\!/) \Psi - m \bar{\Psi}\Psi, \tag{18}$$

where $\mathcal{I}_2$ is a 2×2 unit matrix and the both fields were introduced as a doublet

$$\Psi = \begin{pmatrix} \nu \\ e \end{pmatrix}. \tag{19}$$

The Lagrangian (18) is invariant with respect to the $U(2) = SU(2) \times U(1)$ group of transformations

$$\Psi \rightarrow U \exp(i\alpha)\Psi, \quad U = \exp(i\varphi^a \tau^a) \in SU(2), \quad \exp(i\alpha) \in U(1). \tag{20}$$

The doublet (19) is a member of a two-dimensional vector space. This space is closed with respect to the symmetry transformations. All vectors of this internal vector space represent equivalent physical systems.

The Lagrangian (18) can be rewritten in the following way

$$\mathcal{L} = \bar{\Psi}_L i (\mathcal{I}_2 \otimes \partial\!\!\!/) \Psi_L + \bar{\Psi}_R i (\mathcal{I}_2 \otimes \partial\!\!\!/) \Psi_R - m(\bar{\Psi}_L \Psi_R + \bar{\Psi}_R \Psi_L), \tag{21}$$

where

$$\Psi_{L,R} = \begin{pmatrix} \nu_{L,R} \\ e_{L,R} \end{pmatrix}. \tag{22}$$

If $m = 0$ the symmetry of the Lagrangian gets promoted to the chiral one, $SU(2)_L \times SU(2)_R \times U(1)_L \times U(1)_R$. The transformations of the fields are

$$\Psi_{L,R} \rightarrow U_{L,R} \exp(i\alpha_{L,R})\Psi_{L,R}, \quad U_{L,R} = \exp(i\varphi^a_{L,R}\tau^a) \in SU(2)_{L,R}, \quad \exp(i\alpha_{L,R}) \in U(1)_{L,R}. \tag{23}$$

Note that transformations (20) are the "diagonal" subgroup of $SU(2)_L \times SU(2)_R \times U(1)_L \times U(1)_R$. They are obtained from (23) if $\alpha_L = \alpha_R$ and $\varphi^a_L = \varphi^a_R$.

3 Interaction with a single vector field

In the Lagrangians of Section 2 we would like to introduce an interaction of fermions with a single vector field A_μ without spoiling the Lagrangian symmetries. Let us call the field *photon*. The simplest interaction term fulfilling the space-time symmetries has a structure

$$\bar{\psi} A\!\!\!/ \psi, \tag{24}$$

where $A\!\!\!/ = A_\mu \gamma^\mu$, as usual, and ψ is a Dirac bispinor. What follows is a list of the Lagrangians introduced above and extended by the apropriate interaction term of fermions with a photon:

- The $U(1)$ Lagrangian (g is a coupling constant):

$$\mathcal{L} = \bar{\psi} i \not{\partial} \psi - m_e \bar{\psi}\psi + g\bar{\psi}\not{A}\psi. \tag{25}$$

  Transformation of A:

$$U(1) : \not{A} \to \not{A} \tag{26}$$

- The $U(1)_L \times U(1)_R$ Lagrangian ($g_{L,R}$ are coupling constants):

$$\mathcal{L} = \bar{\psi}_L i \not{\partial} \psi_L + \bar{\psi}_R i \not{\partial} \psi_R + g_L \bar{\psi}_L \not{A} \psi_L + g_R \bar{\psi}_R \not{A} \psi_R. \tag{27}$$

  Transformation of A:

$$U(1)_{L,R} : \not{A} \to \not{A} \tag{28}$$

- The $U(1)_\nu \times U(1)_e$ Lagrangian ($g_{\nu,e}$ are coupling constants):

$$\mathcal{L} = \bar{\nu} i \not{\partial} \nu + \bar{e} i \not{\partial} e - m_\nu \bar{\nu}\nu - m_e \bar{e} e + g_\nu \bar{\nu} \not{A} \nu + g_e \bar{e} \not{A} e. \tag{29}$$

  Transformation of A:

$$U(1)_{\nu,e} : \not{A} \to \not{A} \tag{30}$$

- The $SU(2) \times U(1)$ Lagrangian (g is a coupling constant):

$$\mathcal{L} = \bar{\Psi} i (\mathcal{I}_2 \otimes \not{\partial}) \Psi - m \bar{\Psi}\Psi + g \bar{\Psi} (\mathcal{I}_2 \otimes \not{A}) \Psi. \tag{31}$$

  Transformation of A:

$$U(1) : \not{A} \to \not{A}, \quad SU(2) : \mathcal{I}_2 \otimes \not{A} \to \mathcal{I}_2 \otimes \not{A} \tag{32}$$

- The $SU(2)_L \times SU(2)_R \times U(1)_L \times U(1)_R$ Lagrangian ($g_{L,R}$ are coupling constants):

$$\mathcal{L} = \bar{\Psi}_L i (\mathcal{I}_2 \otimes \not{\partial}) \Psi_L + \bar{\Psi}_R i (\mathcal{I}_2 \otimes \not{\partial}) \Psi_R + g_L \bar{\Psi}_L (\mathcal{I}_2 \otimes \not{A}) \Psi_L + g_R \bar{\Psi}_R (\mathcal{I}_2 \otimes \not{A}) \Psi_R. \tag{33}$$

  Transformation of A:

$$U(1)_{L,R} : \not{A} \to \not{A}, \quad SU(2)_{L,R} : \mathcal{I}_2 \otimes \not{A} \to \mathcal{I}_2 \otimes \not{A} \tag{34}$$

In other words, the field $\not{A}$ is always a singlet with respect to the groups in concern; it stays invariant. Regarding the standard kinetic and mass terms for the field A_μ:

$$F^{\mu\nu} F_{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{35}$$

$$m_A^2 A^\mu A_\mu, \tag{36}$$

respectively, they are invariant, and thus acceptable, for all cases under consideration.