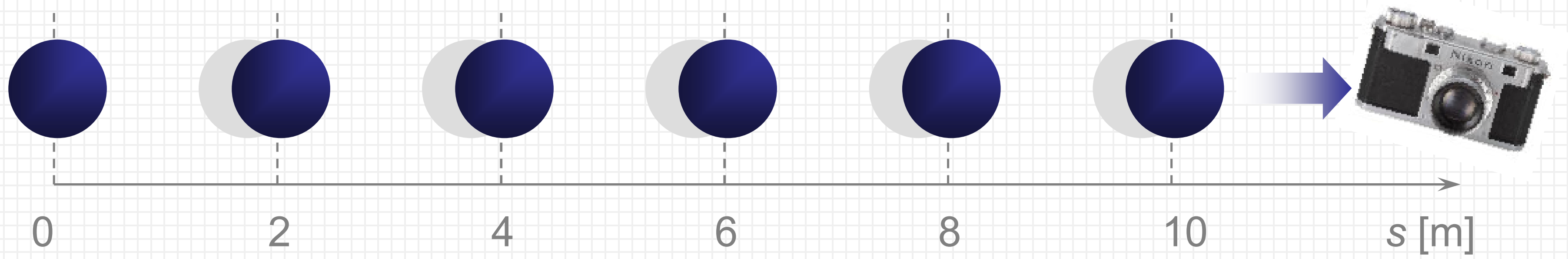
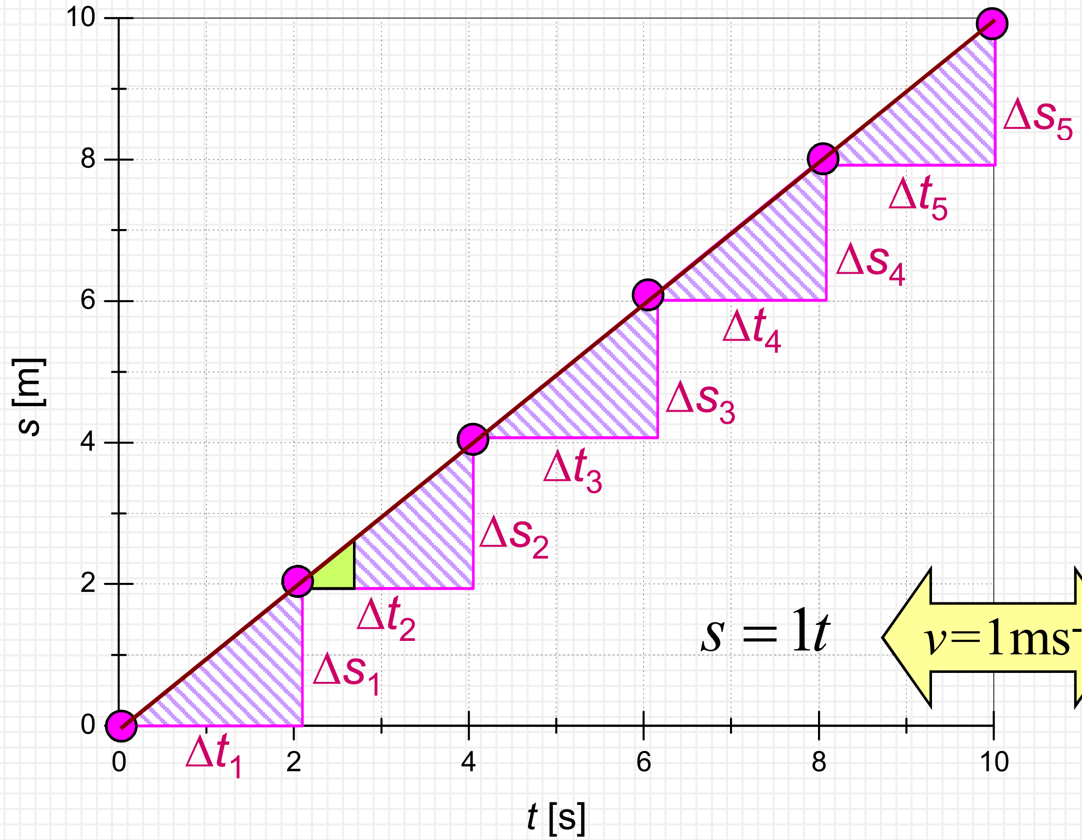
Kinematika hmotného bodu

Rovnomerný pohyb, rovnomerne zrýchlený pohyb.
Rýchlosť ako derivácia, dráha ako integrál.
Základy diferenciálneho počtu.
Polohový vektor, vektor rýchlosti, vektor zrýchlenia.
Pohyb po kružnici, uhlová rýchlosť, uhlové zrýchlenie.
Perióda a frekvencia.
Tangenciálne a normálové zrýchlenie.
Klasifikácia pohybov.



Δt = 2s



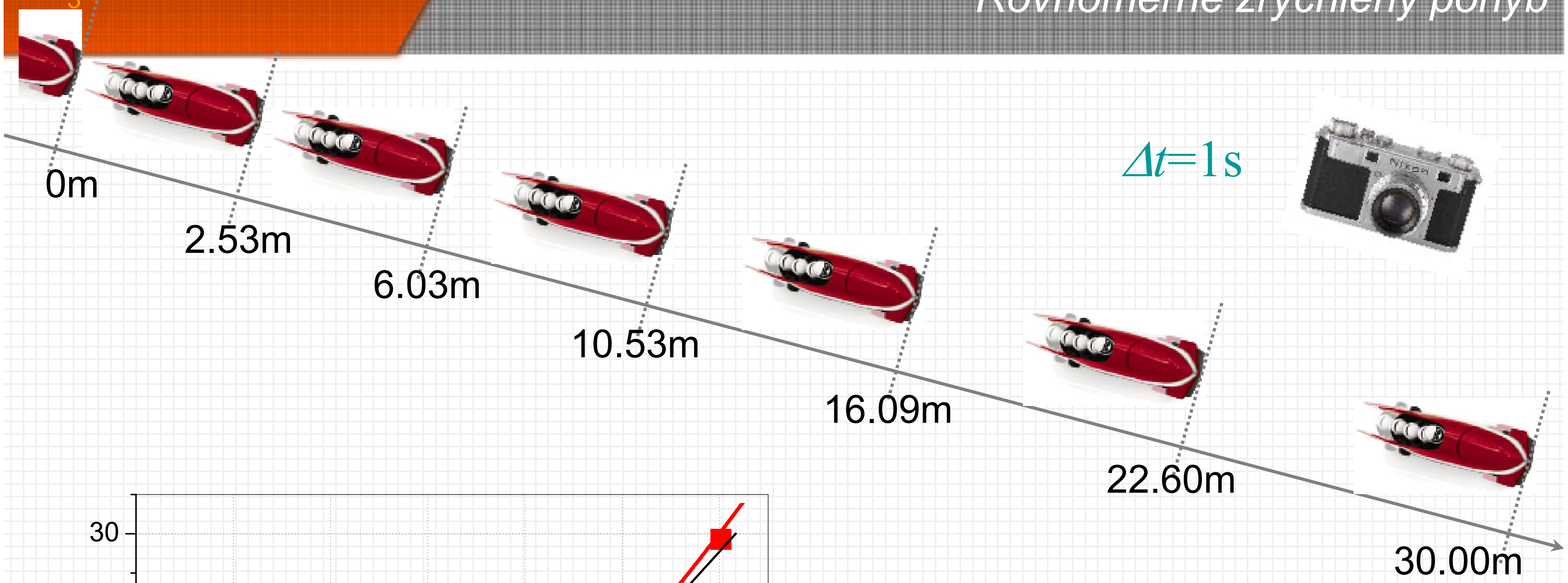
$$\frac{\Delta s_1}{\Delta t_1} = \frac{\Delta s_2}{\Delta t_2} = \ldots = \frac{\Delta s_5}{\Delta t_5} = v$$

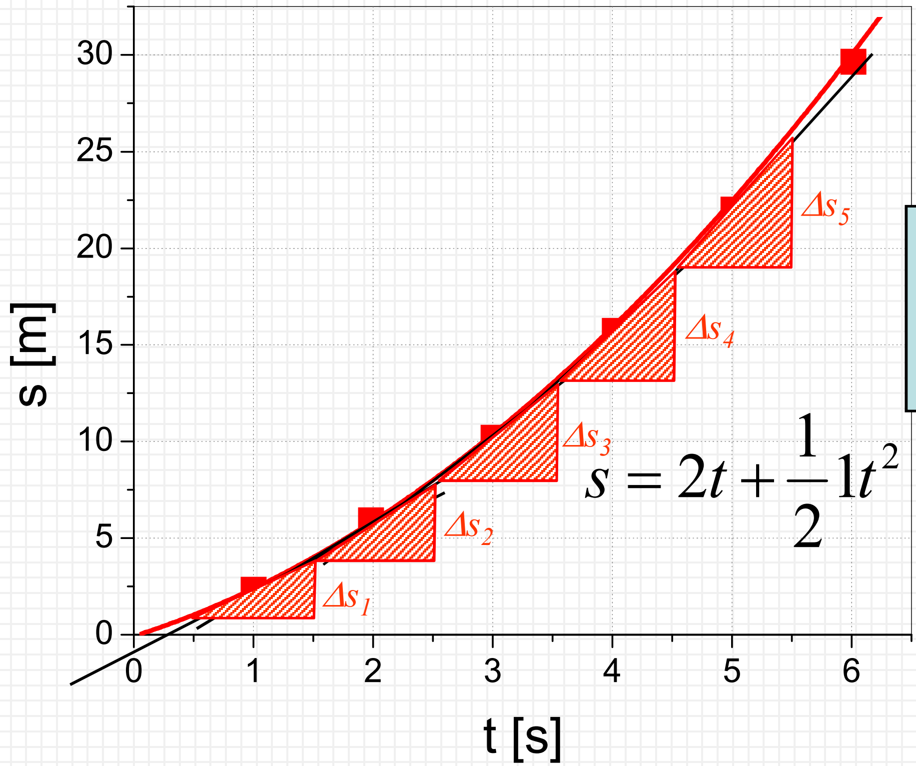
$$\Delta t \to 0$$

$$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{\mathrm{d}s}{\mathrm{d}t} = v$$

$$s = 1t \quad \Longleftrightarrow \quad v = 1\,\mathrm{ms}^{-1} \quad \Longleftrightarrow \quad s = vt$$

	t [s]	s [m]	
	0	0	
Δt_1			Δs_1
	2	2	
Δt_2			Δs_2
	4	4	
Δt_3			Δs_3
	6	6	
Δt_4			Δs_4
	8	8	
Δt_5			Δs_5
	10	10	





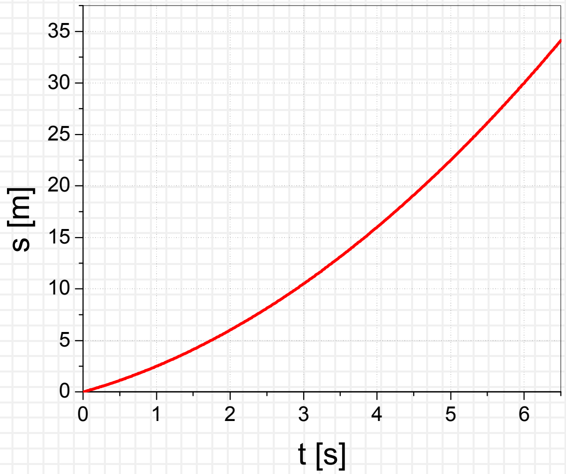
$$s = 2t + \frac{1}{2}1t^2$$

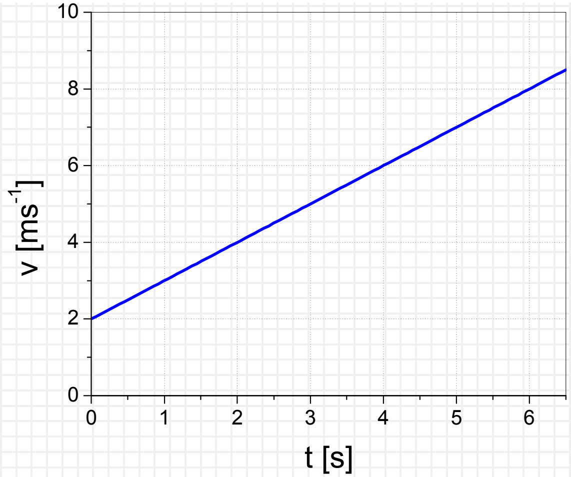
$v_0 = 2\ \mathrm{ms}^{-1}$
$a = 1\ \mathrm{ms}^{-2}$

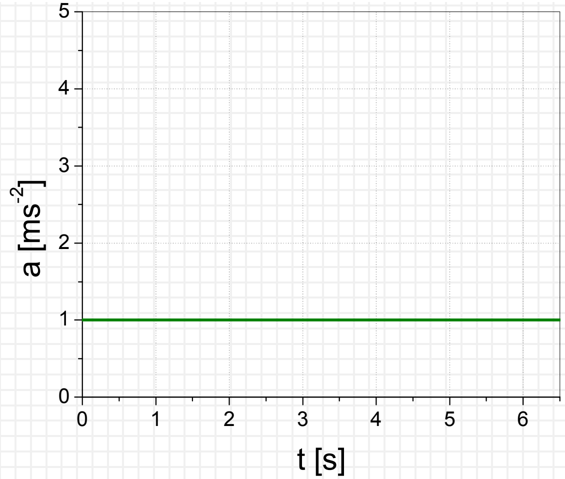
$$\Leftrightarrow$$

$$s = v_0 t + \frac{1}{2}at^2$$

t [s]	s [m]
1.0	2.53
2.0	6.03
3.0	10.53
4.0	16.09
5.0	22.60
6.0	30.00







$$s = 2t + \frac{1}{2}1t^2$$

$$v = 2 + t$$

$$a = 1$$

$$\boxed{v = \frac{\mathrm{d}s}{\mathrm{d}t}}$$

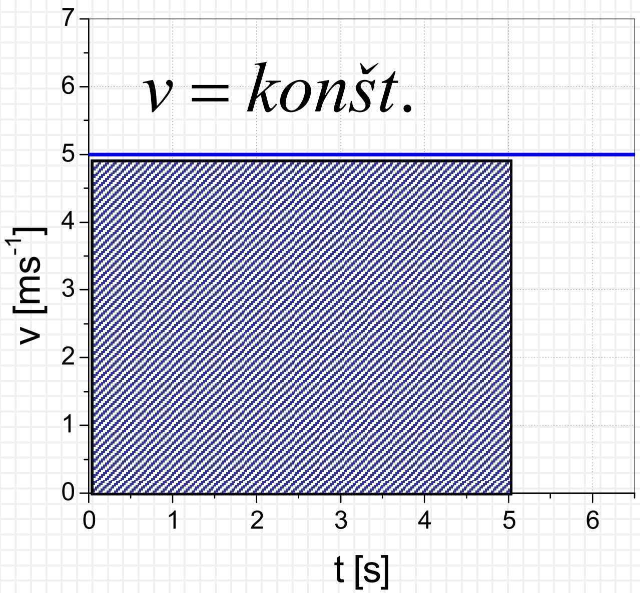
$$\boxed{a = \frac{\mathrm{d}v}{\mathrm{d}t}}$$

$$s = v_0 t + \frac{1}{2}at^2$$

$$v = v_0 + at$$

$v_0 = 2\mathrm{ms}^{-1}$
$a = 1\mathrm{ms}^{-2}$

Rovnomerný pohyb

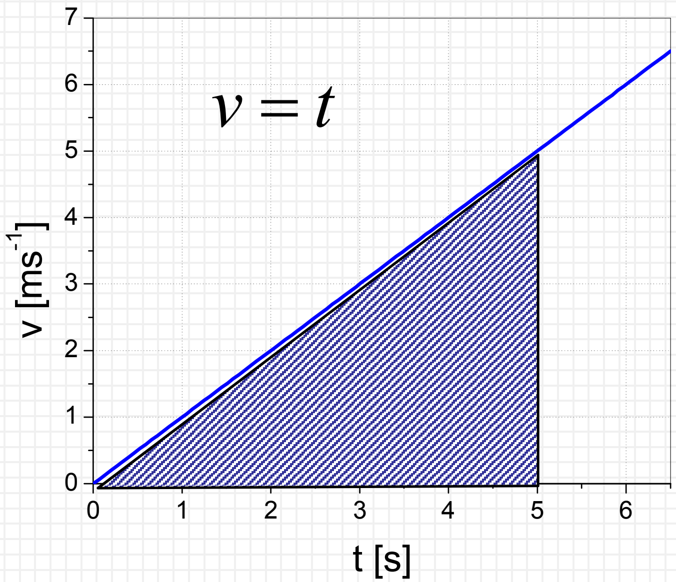


$$s = vt$$

Akú dráhu prejde za čas 5s?

$$s = 5\mathrm{ms}^{-1}5\mathrm{s} = 25\mathrm{m}$$

Rovnomerne zrýchlený pohyb



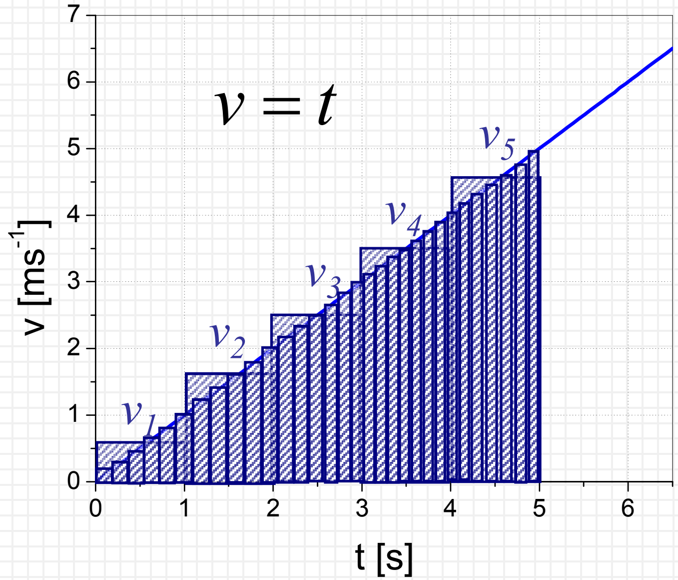
$$s = \frac{vt}{2}$$

Akú dráhu prejde za čas 5s?

$$s = \frac{5\mathrm{ms}^{-1}5\mathrm{s}}{2} = 12{,}5\mathrm{m}$$

Dráha zodpovedá ploche pod krivkou ...

Rovnomerne zrýchlený pohyb



$$s = \sum_{i=1}^{5} v_i \Delta t$$

veľká nepresnosť, zmenšenie Δt

Ešte vylepšenie $\Delta t \to 0$

$$s = \lim_{\Delta t \to 0} \sum_i v_i \Delta t$$

$$\int v \mathrm{d}t$$

Integrál z funkcie

$$(x^n)' = nx^{n-1}$$

$$(\sin(x))' = \cos(x)$$

$$(\cos(x))' = -\sin(x)$$

$$(e^x)' = e^x$$

$$(\ln(x))' = \frac{1}{x}$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, (n \neq -1)$$

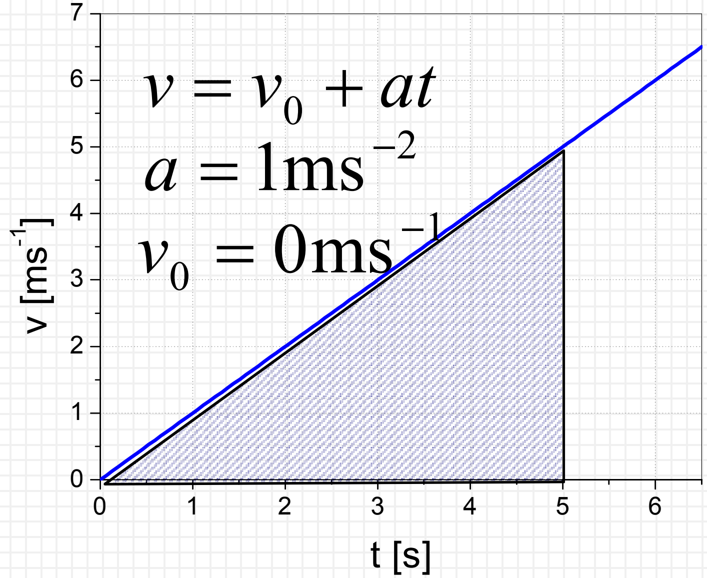
$$\int \sin(x) dx = -\cos(x) + C$$

$$\int \cos(x) dx = \sin(x) + C$$

$$\int e^x dx = e^x + C$$

$$\int \frac{1}{x} dx = \ln(x) + C$$

Výpočet dráhy rovnomerne zrýchleného pohybu



$$v = v_0 + at$$
$$a = 1\mathrm{ms}^{-2}$$
$$v_0 = 0\mathrm{ms}^{-1}$$

$$\boxed{v = \frac{ds}{dt}} \Rightarrow vdt = ds \Rightarrow \int_0^t vdt = \int_{s_0}^{s} ds$$

$$\int_0^t (v_0 + at)dt = \int_{s_0}^{s} ds$$

$$v_0 \int_0^t dt + a \int_0^t tdt = \int_{s_0}^{s} ds$$

$$v_0 [t]_0^t + a \left[\frac{t^2}{2}\right]_0^t = [s]_{s_0}^{s}$$

$$v_0 t + \frac{1}{2}at^2 = s - s_0$$

$$v = at \Rightarrow a = \frac{v}{t}$$

$$v_0 = 0$$
$$s_0 = 0$$

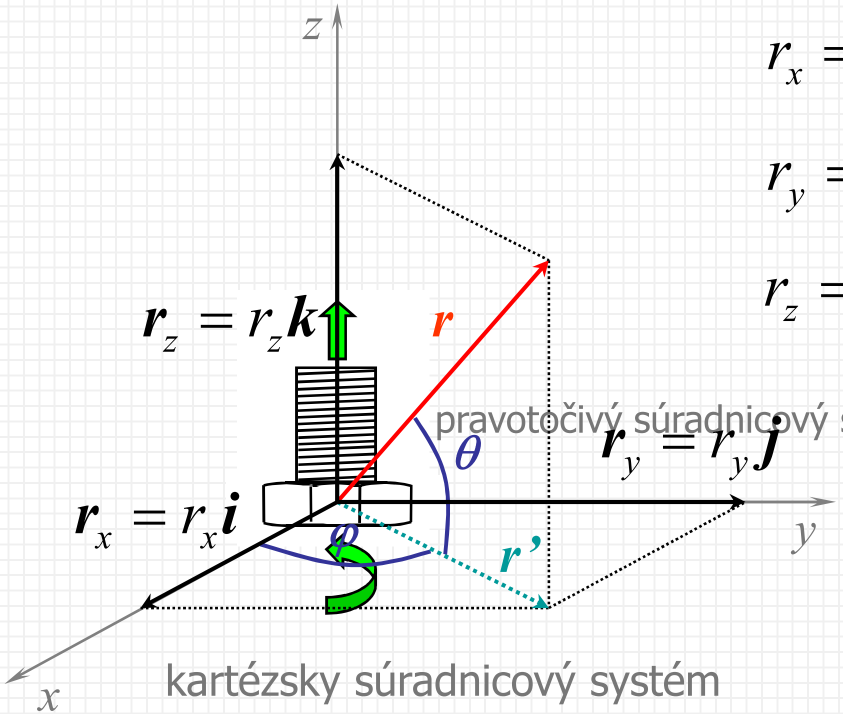
$$\boxed{s = \frac{1}{2}vt} \Leftarrow s = \frac{1}{2}\frac{v}{t}t^2 \Leftarrow s = \frac{1}{2}at^2 \Leftarrow s = s_0 + v_0 t + \frac{1}{2}at^2$$

sférický súradnicový systém

$$r_x = |\boldsymbol{r}| \cos\theta \cos\varphi$$

$$r_y = |\boldsymbol{r}| \cos\theta \sin\varphi$$

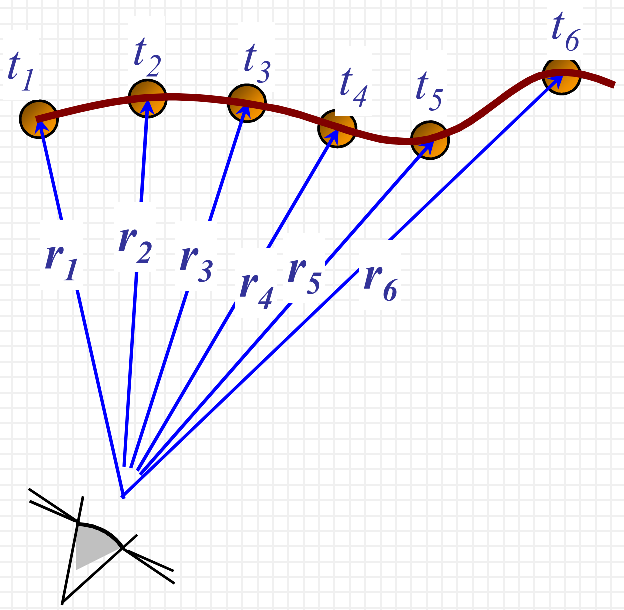
$$r_z = |\boldsymbol{r}| \sin\theta$$

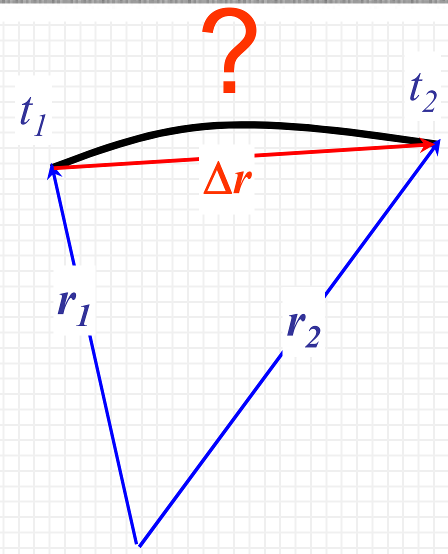


kartézsky súradnicový systém

$$\boldsymbol{r} = r_x \boldsymbol{i} + r_y \boldsymbol{j} + r_z \boldsymbol{k}$$

Diferenciál polohového vektora



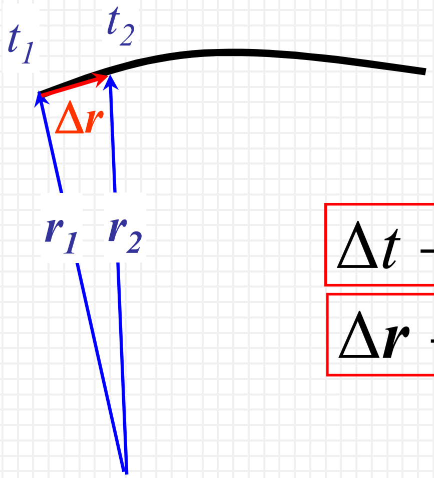


$$\boldsymbol{r}_1 + \Delta\boldsymbol{r} = \boldsymbol{r}_2$$

$$\boxed{\Delta\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1}$$

$$\boxed{\Delta t = t_2 - t_1}$$

Zmenšovanie časového intervalu⇨lepšie popísanie krivky



$$\boxed{\Delta t \to 0}$$

$$\boxed{\Delta\boldsymbol{r} \to \mathrm{d}\boldsymbol{r}}$$

$$\boldsymbol{v} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{r}}{\Delta t} \Rightarrow \boxed{\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}t}\boldsymbol{\tau}}$$ okamžitá rýchlosť

[ms^{-1}] … jednotka rýchlosti

Priemerná (stredná) rýchlosť

$$\boldsymbol{v}_p = \frac{\Delta \boldsymbol{r}}{\Delta t} \Rightarrow v_p = \frac{\Delta x}{\Delta t}$$ (priamočiary pohyb)

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \frac{\mathrm{d}(r_x\boldsymbol{i} + r_y\boldsymbol{j} + r_z\boldsymbol{k})}{\mathrm{d}t} = \boldsymbol{i}\frac{\mathrm{d}r_x}{\mathrm{d}t} + \boldsymbol{j}\frac{\mathrm{d}r_y}{\mathrm{d}t} + \boldsymbol{k}\frac{\mathrm{d}r_z}{\mathrm{d}t} = \boldsymbol{i}v_x + \boldsymbol{j}v_y + \boldsymbol{k}v_z$$

$$\Downarrow$$

$$|\boldsymbol{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$ pre tri zložky rýchlosti

$$|\boldsymbol{v}| = \sqrt{v_x^2 + v_y^2}$$ pre dve zložky rýchlosti

Zrýchlenie, vektor zrýchlenia

$$\boldsymbol{a} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{v}}{\Delta t}$$

$$\boxed{\boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{\mathrm{d}\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)}{\mathrm{d}t} = \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}}$$

[ms^{-2}] … jednotka zrýchlenia

$$\boldsymbol{a} = \boldsymbol{i}\frac{\mathrm{d}v_x}{\mathrm{d}t} + \boldsymbol{j}\frac{\mathrm{d}v_y}{\mathrm{d}t} + \boldsymbol{k}\frac{\mathrm{d}v_z}{\mathrm{d}t} = \boldsymbol{i}a_x + \boldsymbol{j}a_y + \boldsymbol{k}a_z$$

$$|\boldsymbol{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$ pre tri zložky zrýchlenia

$$|\boldsymbol{a}| = \sqrt{a_x^2 + a_y^2}$$ pre dve zložky zrýchlenia

$$\boldsymbol{v} = v\boldsymbol{\tau}$$

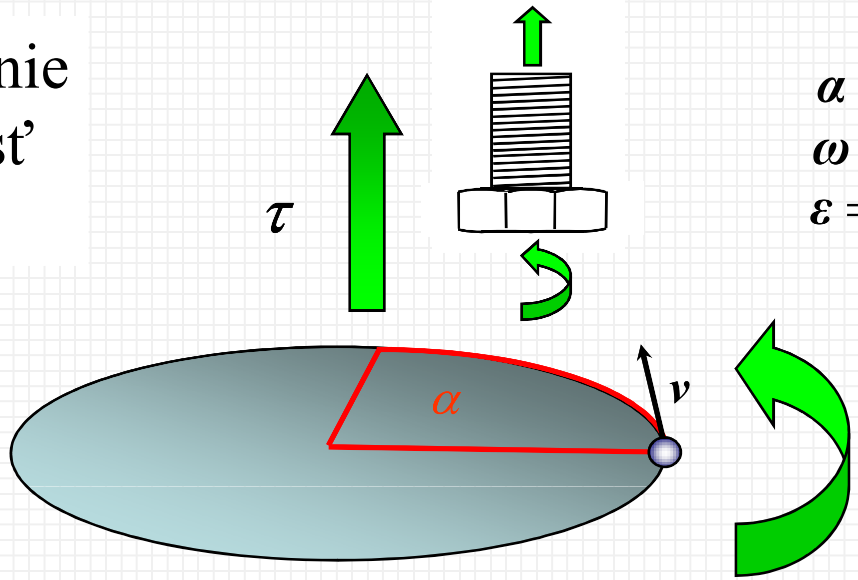
$v \rightarrow$ =konšt. Rovnomerný pohyb

$v \rightarrow$ ≠konšt. Nerovnomerný pohyb

$\tau \rightarrow$ =konšt. Priamočiary pohyb

$\tau \rightarrow$ ≠konšt. Krivočiary pohyb

ε ... uhlové zrýchlenie
ω ... uhlová rýchlosť
α ... uhol



$$\boldsymbol{\alpha} = \alpha\boldsymbol{\tau}$$
$$\boldsymbol{\omega} = \omega\boldsymbol{\tau}$$
$$\boldsymbol{\varepsilon} = \varepsilon\boldsymbol{\tau}$$

$$\boxed{\boldsymbol{\omega} = \frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}t}} \Longleftrightarrow \boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}$$

$$\boxed{\boldsymbol{\varepsilon} = \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} = \frac{\mathrm{d}^2\boldsymbol{\alpha}}{\mathrm{d}t^2}} \Longleftrightarrow \boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}$$

Rovnomerný pohyb po kružnici

$$\boxed{\omega = \omega_0 = konšt.}$$

$$\omega_0 \mathrm{d}t = \mathrm{d}\alpha$$

$$\int_0^t \omega_0 \mathrm{d}t = \int_{\alpha_0}^{\alpha} \mathrm{d}\alpha$$

$$\omega_0 \int_0^t \mathrm{d}t = \int_{\alpha_0}^{\alpha} \mathrm{d}\alpha$$

$$\omega_0 [t]_0^t = [\alpha]_{\alpha_0}^{\alpha}$$

$$\omega_0 t = \alpha - \alpha_0$$

$$\boxed{\omega_0 t = \alpha}$$

Rovnomerne zrýchlený pohyb po kružnici

$$\boxed{\omega \neq konšt.} \qquad \boxed{\omega = \omega_0 + \varepsilon t}$$

$$\boxed{\omega = \frac{\mathrm{d}\alpha}{\mathrm{d}t} \Rightarrow \omega \mathrm{d}t = \mathrm{d}\alpha}$$

$$(\omega_0 + \varepsilon t)\mathrm{d}t = \mathrm{d}\alpha$$

$$\int_0^t (\omega_0 + \varepsilon t)\mathrm{d}t = \int_{\alpha_0}^{\alpha} \mathrm{d}\alpha$$

$$\omega_0 \int_0^t \mathrm{d}t + \varepsilon \int_0^t t\mathrm{d}t = \int_{\alpha_0}^{\alpha} \mathrm{d}\alpha$$

$$\omega_0 [t]_0^t + \varepsilon \left[\frac{t^2}{2}\right]_0^t = [\alpha]_{\alpha_0}^{\alpha}$$

$$\omega_0 t + \frac{1}{2}\varepsilon t^2 = \alpha - \alpha_0$$

v čase t=0

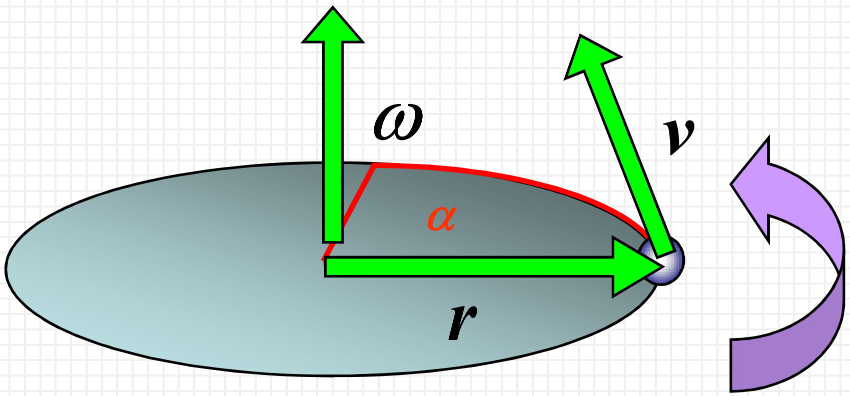
$$\alpha_0 = 0$$

$$\boxed{\omega_0 t + \frac{1}{2}\varepsilon t^2 = \alpha}$$

Analógia vzťahov pre rovnomerný pohyb a po kružnici

Pohyb po kružnici	Priamočiary pohyb
α	s $(\boldsymbol{r})$
ω	$\boldsymbol{v}$
ε	$\boldsymbol{a}$
$\boxed{\boldsymbol{\omega} = \dfrac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}t}}$	$\boxed{\boldsymbol{v} = \dfrac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}}$
$\boxed{\boldsymbol{\varepsilon} = \dfrac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}}$	$\boxed{\boldsymbol{a} = \dfrac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}}$
$\boxed{\omega = \omega_0 + \varepsilon t}$	$\boxed{v = v_0 + at}$
$\boxed{\alpha = \omega_0 t + \dfrac{1}{2}\varepsilon t^2}$	$\boxed{s = v_0 t + \dfrac{1}{2}at^2}$

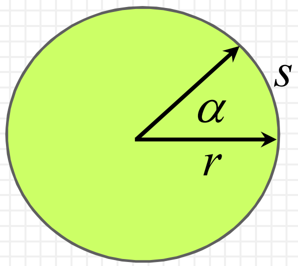
Vzťah medzi obvodovou a uhlovou rýchlosťou



v ... obvodová rýchlosť
ω ... uhlová rýchlosť
r ... polomer

Vychádzame zo vzťahu:

$$\boxed{v = \frac{\mathrm{d}s}{\mathrm{d}t}}$$



$s = r\alpha$ ... kruhový oblúk

$\mathrm{d}s = r\mathrm{d}\alpha$ ... malý element

$$\boxed{v = r\frac{\mathrm{d}\alpha}{\mathrm{d}t}} \Rightarrow \boxed{v = r\omega}$$

Ak uvážime smer vektorov tak platí:

$$\boxed{\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}}$$

Len pre rovnomerný pohyb po kružnici!!!!!

Perióda kruhového pohybu (T)

(vyjadruje čas, za ktorý hm. b. vykonal rovnomerným pohybom jeden obeh po kružnici)

$$T = \frac{s}{v} = \frac{2\pi r}{v} = \frac{2\pi \cancel{r}}{\cancel{r}\omega} \Rightarrow \boxed{T = \frac{2\pi}{\omega}}$$

jednotka … [s]

Frekvencia kruhového pohybu (f)

(vyjadruje počet obehov za jednotku času)

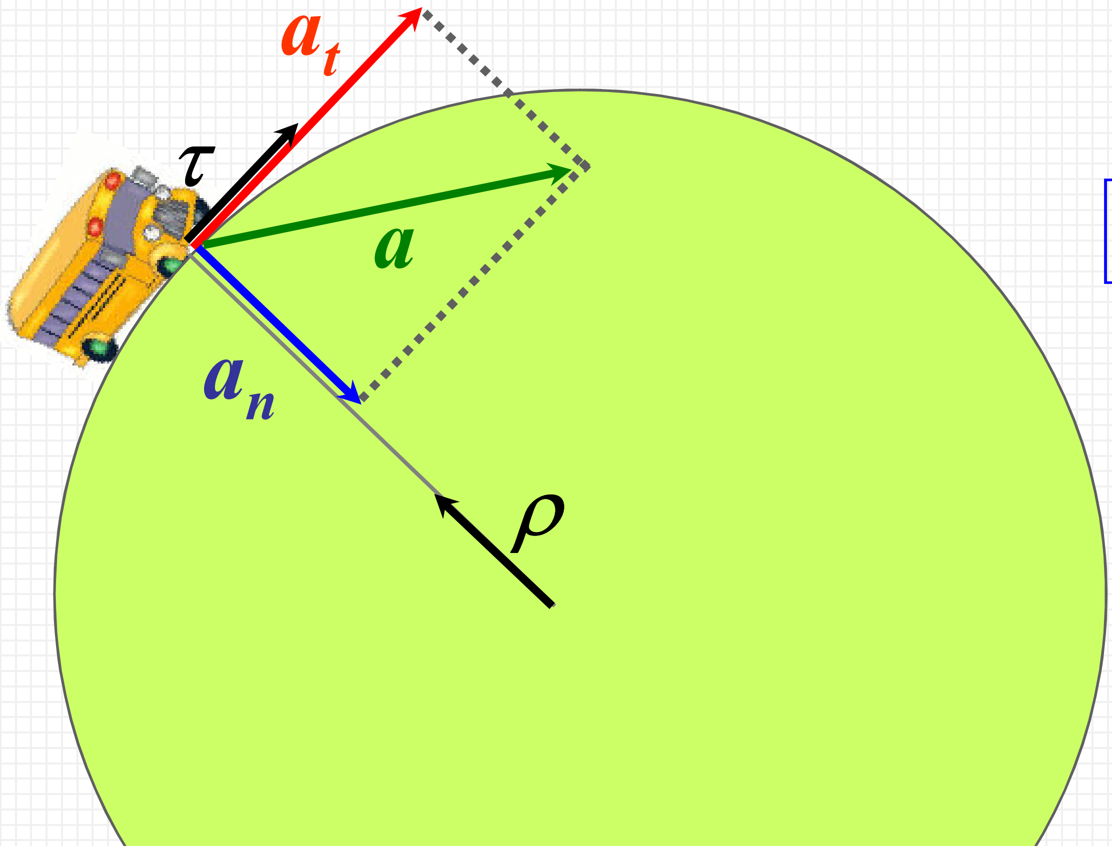
$$f = \frac{1}{T} \Rightarrow \boxed{f = \frac{\omega}{2\pi}}$$

jednotka … [Hz] (Hertz)

a_t ... tangenciálne zrýchlenie
(vyjadruje zmenu veľkosti rýchlosti)

a_n ... normálové zrýchlenie
(vyjadruje zmenu smeru rýchlosti)

Príklady z praxe:
autobus v zákrute
odletujúce iskry z brúsky
centrifúga, kolotoč



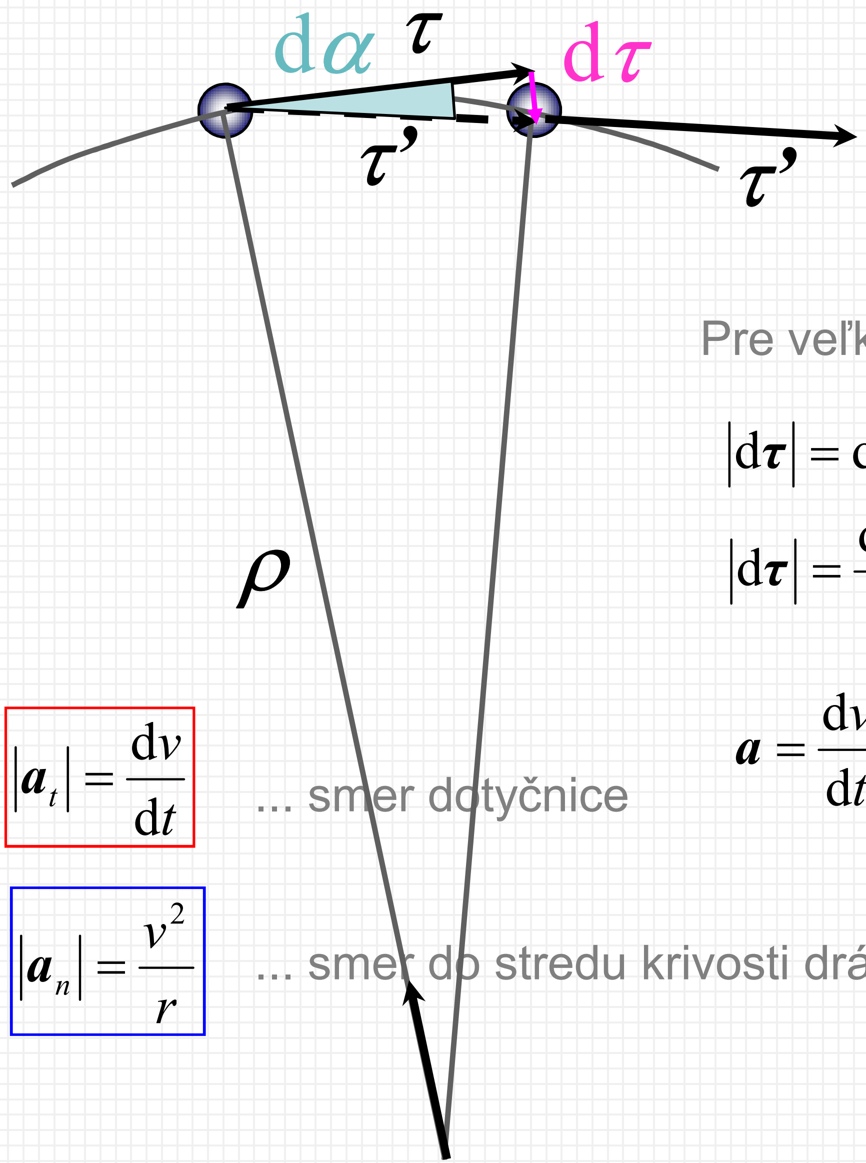
$$\boxed{\boldsymbol{a} = \boldsymbol{a}_t + \boldsymbol{a}_n}$$

$$\boldsymbol{a}_t = a_t \boldsymbol{\tau}$$

$$\boxed{|\boldsymbol{a}| = \sqrt{a_t^2 + a_n^2}}$$

$$\boldsymbol{a}_n = -a_n \boldsymbol{\rho}$$

Tangenciálne a normálové zrýchlenie II



$$\boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(v\boldsymbol{\tau}) = \underbrace{\frac{\mathrm{d}v}{\mathrm{d}t}\boldsymbol{\tau}}_{\boldsymbol{a}_t} + \underbrace{v\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}t}}_{\boldsymbol{a}_n}$$

Pre veľkosť $d\tau$ platí:

$$|\mathrm{d}\boldsymbol{\tau}| = \mathrm{d}\alpha$$

$$|\mathrm{d}\boldsymbol{\tau}| = \frac{\mathrm{d}s}{r}$$

uvážime aj smer: $\boxed{\mathrm{d}\boldsymbol{\tau} = -\frac{\mathrm{d}s}{r}\boldsymbol{\rho}}$

$$\boldsymbol{a} = \frac{\mathrm{d}v}{\mathrm{d}t}\boldsymbol{\tau} - v\frac{1}{\mathrm{d}t}\frac{\mathrm{d}s}{r}\boldsymbol{\rho} \Rightarrow \boldsymbol{a} = \underbrace{\frac{\mathrm{d}v}{\mathrm{d}t}\boldsymbol{\tau}}_{\boldsymbol{a}_t} - \underbrace{\frac{v^2}{r}\boldsymbol{\rho}}_{\boldsymbol{a}_n}$$

$\boxed{|\boldsymbol{a}_t| = \frac{\mathrm{d}v}{\mathrm{d}t}}$ ... smer dotyčnice

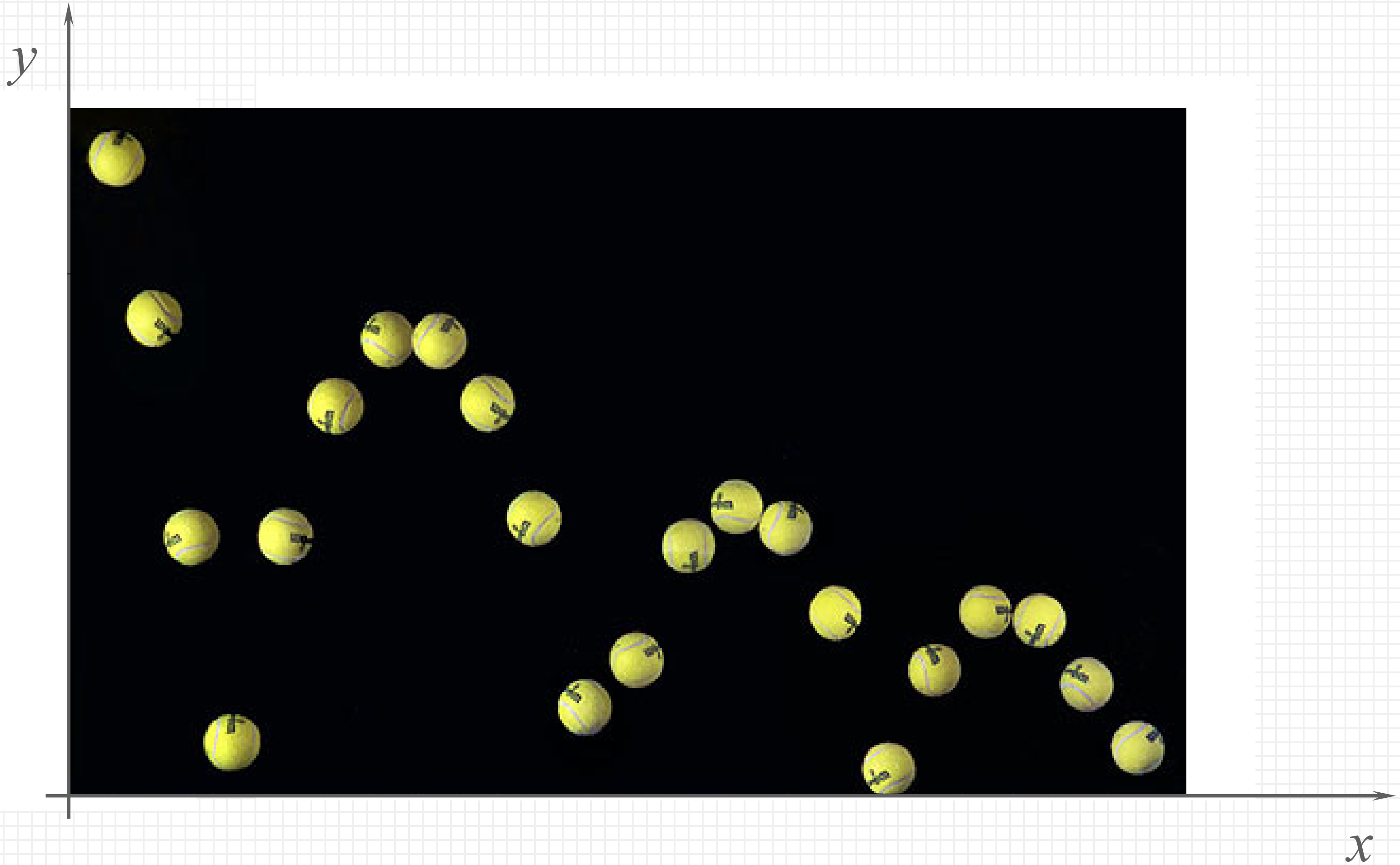
$\boxed{|\boldsymbol{a}_n| = \frac{v^2}{r}}$ ... smer do stredu krivosti dráhy

$\boxed{F_g = m\boldsymbol{g}}$ v gravitačnom poli sa prejavujú účinky tiaže

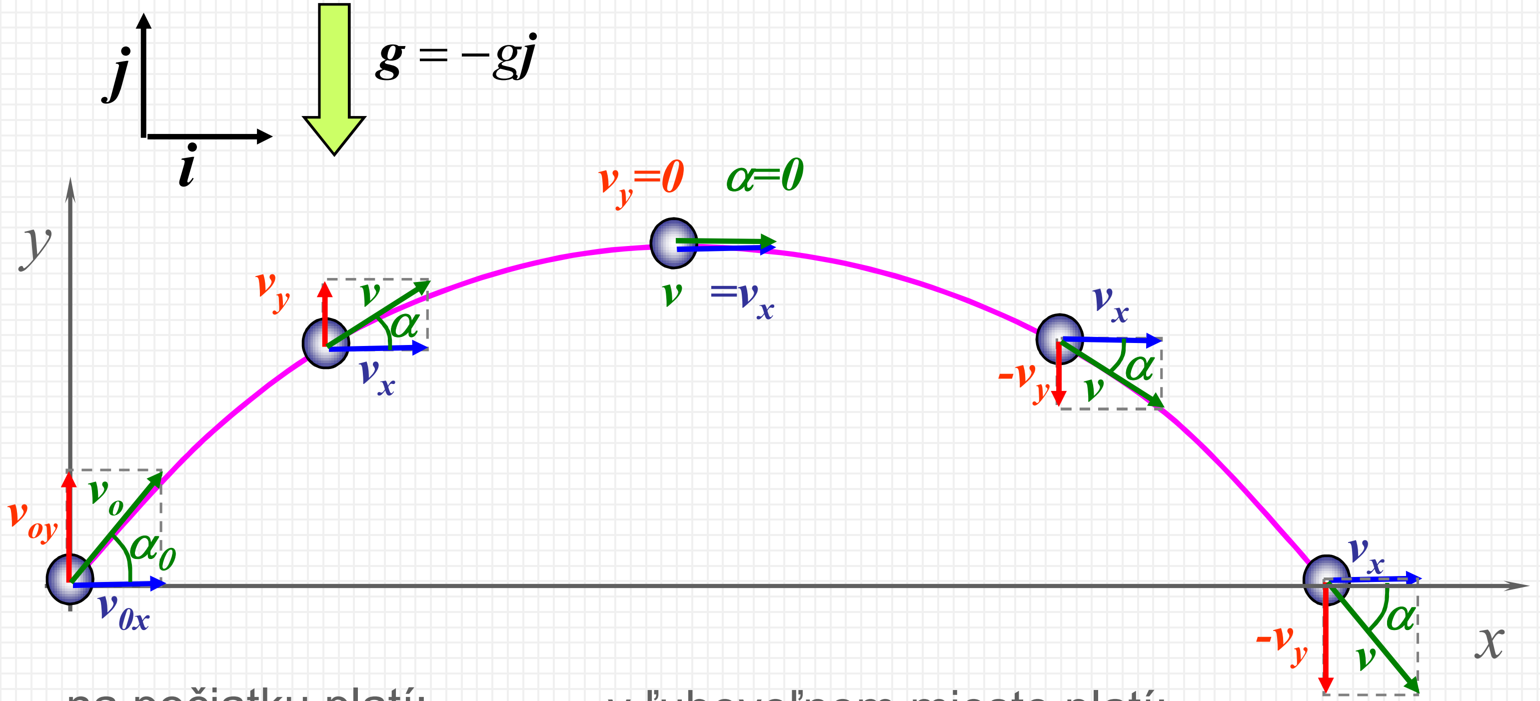
$\boxed{\boldsymbol{a} \leftrightarrow \boldsymbol{g}}$ analógia so zrýchleným pohybom

$$v = v_0 + gt$$

$$s = v_0 t + \frac{1}{2} g t^2$$



Šikmý vrh – počiatočné podmienky, rýchlosť



na počiatku platí:

$$\boldsymbol{v}_0 = v_{0x}\boldsymbol{i} + v_{0y}\boldsymbol{j}$$

$$v_{ox} = v_0 \cos\alpha_0$$

$$v_{oy} = v_0 \sin\alpha_0$$

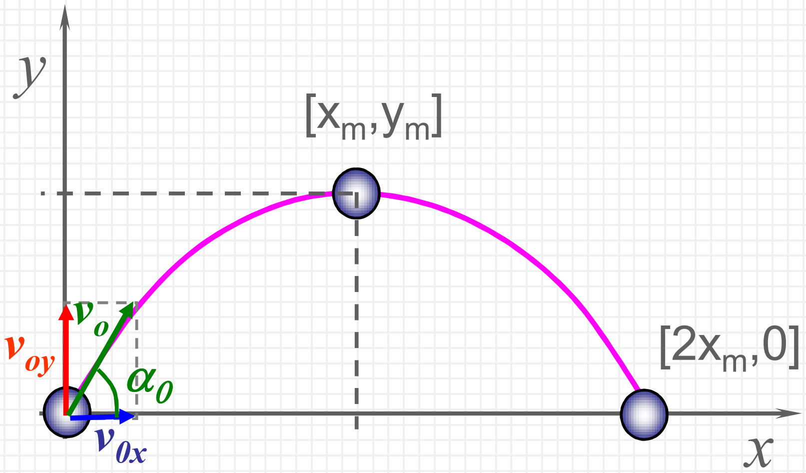
v ľubovoľnom mieste platí:

$$\boldsymbol{v} = v_x\boldsymbol{i} + v_y\boldsymbol{j}$$

$$v_x = v_0 \cos\alpha_0$$

$$v_y = v_{0y} - gt = v_0 \sin\alpha_0 - gt$$

Šikmý vrh – rovnice dráhy, max. výška



Pre polohu hm. b. platí:

$$dx = v_x dt$$

$$x = \int v_o \cos\alpha_0 dt = v_o \cos\alpha_0 \int dt$$

$$\boxed{x = v_0 \cos(\alpha_0) t}$$

Analogicky získame:

$$\boxed{y = v_0 \sin(\alpha_0) t - \frac{1}{2} g t^2}$$

rovnice dráhy pohybu

Určenie max. výšky:

Vyjadríme t z rovnice dráhy pohybu pre x a dosadíme za y:

$$y = v_0 \sin\alpha_0 \frac{x}{v_0 \cos\alpha_0} - \frac{1}{2} g \frac{x^2}{v_0^2 \cos^2\alpha_0} \Rightarrow y = x\,tg\alpha_0 - \frac{x^2 g}{2 v_0^2 \cos^2\alpha_0}$$

V maxime krivky platí:

$$\frac{dy}{dx} = 0 \Rightarrow tg\alpha_0 - \frac{g}{v_0^2 \cos^2\alpha_0} x_m = 0 \Rightarrow x_m = \frac{v_0^2\, tg\alpha_0 \cos^2\alpha_0}{g}$$

$$x_m = \frac{v_0^2 tg\alpha_0 \cos^2 \alpha_0}{g} \Rightarrow x_m = \frac{v_0^2 \sin\alpha_0 \cos\alpha_0}{g}$$

$$\boxed{x_m = \frac{v_0^2 \sin 2\alpha_0}{2g}}$$

Dosadením pre y_m dostávame:

$$y_m = tg\alpha_0 \frac{v_0^2 \sin 2\alpha_0}{2g} - \frac{g}{2v_0^2 \cos^2 \alpha_0} \frac{v_0^4 \sin^2 2\alpha_0}{4g^2}$$

$$\boxed{y_m = \frac{v_0^2 \sin^2 \alpha_0}{2g}}$$

súradnice vrcholu paraboly šikmého vrhu

Pre dĺžku platí:

$$\boxed{d = 2x_m = \frac{2v_0^2 \sin 2\alpha_0}{2g} = \frac{v_0^2 \sin 2\alpha_0}{g}}$$

$$\sin\alpha \cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

$$\sin 2\alpha = \sin\alpha\cos\alpha + \sin\alpha\cos\alpha = 2\sin\alpha\cos\alpha$$

Vodorovný vrh, zvislý vrh nahor, nadol, voľný pád

Vychádzame zo vzťahov pre šikmý vrh:

$$v_x = v_0 \cos\alpha_0 \qquad x = v_0 \cos(\alpha) t$$

$$v_y = v_0 \sin\alpha_0 - gt \qquad y = v_0 t \sin\alpha_0 - \frac{1}{2} g t^2$$

Rozdelenie vrhov na základe: α_0 v_0

Vodorovný vrh: $\alpha_0 = 0$

$$v_x = v_0 \qquad x = v_0 t$$

$$v_y = -gt \qquad y = -\frac{1}{2} g t^2$$

Zvislý vrh nahor: $\alpha_0 = 90^o$

$$v_x = 0 \qquad x = 0$$

$$v_y = v_0 - gt \qquad y = v_0 t - \frac{1}{2} g t^2$$

Zvislý vrh nadol: $\alpha_0 = -90^o$

$$v_x = 0 \qquad x = 0$$

$$v_y = -v_0 - gt \qquad y = -v_0 t - \frac{1}{2} g t^2$$

Voľný pád: $\alpha_0 = -90^o$ $v_0 = 0$

$$v_x = 0 \qquad x = 0$$

$$v_y = -gt \qquad y = -\frac{1}{2} g t^2$$